Approximation Algorithms

Thomas Rothvoß

Institute of Mathematics
EPFL, Lausanne
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Part 1
Introduction

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Why approximation algorithms?

Task: Solve **NP**-hard optimization problem A
$\rightarrow$ no efficient algorithm (unless $\mathbf{NP} = \mathbf{P}$)

Possible approaches:

- exponential time algorithms $\rightarrow$ some theory but too slow and no lower bounds
- heuristic $\rightarrow$ fast, easy but no guarantee, not much theory
- approximation algorithms $\rightarrow$ rich theory in many cases good lower bounds

Running times: $n =$ number of objects in instance, B biggest appearing number, $\varepsilon > 0$ constant

- exponential: $2^n, n \cdot B$
- polynomial: $n^2, n^{100}, n \cdot \log B, n \cdot 2^{1/\varepsilon}, n^{O(1/\varepsilon)^{O(1/\varepsilon)}}$

Basic definitions

Definition

Let Π be an optimization problem and I is instance for A. Then $OPT_\Pi(I)$ is the value of the optimum solution.

Definition

Let $\alpha \geq 1$. A is an α-approximation algorithm for a minimization problem Π if

$$A(I) \leq \alpha \cdot OPT_\Pi(I) \quad \forall \text{ instances } I$$

where $A(I)$ is the value of the solution, that A returns for I.

- Typical values for α: $1.5, 2, O(1), O(\log n)$
- Usually we omit Π and I in $OPT_\Pi(I)$
- For a maximization problem: $A(I) \geq \frac{1}{\alpha} \cdot OPT_\Pi(I)$
- Attention: Sometimes in literature $\alpha < 1$ for maximization problems. For example $\frac{1}{2}$-apx means $A(I) \geq \frac{1}{2} OPT_\Pi(I)$

Definition PTAS

Definition

A_ε is a polynomial time approximation scheme (PTAS) for a minimization problem Π if

$$A_\varepsilon(I) \leq (1+\varepsilon) \cdot OPT(I) \quad \forall \text{ instances } I$$

and for every fixed $\varepsilon > 0$, the running time of A_ε is polynomial in the input size.

Typical running times: $O(n/\varepsilon), 2^{1/\varepsilon} n^2 \log^2(B), n^{O(1/\varepsilon)^{O(1/\varepsilon)}}$

Definition FPTAS

Definition

A_ε is a fully polynomial time approximation scheme (FPTAS) for a minimization problem Π if for every $\varepsilon > 0$

$$A_\varepsilon(I) \leq (1+\varepsilon) \cdot OPT(I) \quad \forall \text{ instances } I$$

and the running time of A_ε is polynomial in the input size <u>and</u> $1/\varepsilon$.

- Typical running time: $O(n^3/\varepsilon^2)$

Part 2
Steiner tree

Source: *Approximation Algorithms* (Vazirani, Springer Press)

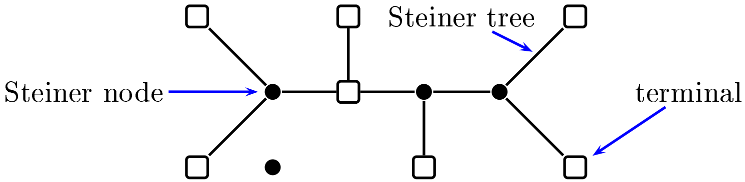
Steiner Tree

Problem: Steiner tree

- Given: Undirected graph $G = (V, E)$, metric cost function $c : E \to \mathbb{Q}_+$, terminals $R \subseteq V$
- Find: Minimum cost tree T connecting all terminals R:

$$OPT = \min\{c(T) \mid T \text{ spans } R\}$$

- $c(T) := \sum_{e \in T} c_e$
- metric: $\forall u, v, w \in V : c_{uw} \leq c_{uv} + c_{vw}$ (triangle inequality)



Steiner tree (2)

Fact

If $R = V$, then Steiner Tree is just the Minimum Spanning Tree Problem which can be solved optimally by picking greedily the cheapest edges (without closing a cycle).

Algorithm:

(1) Compute the minimum spanning tree T on R

(2) Return T

Theorem

The algorithm gives a 2-approximation.

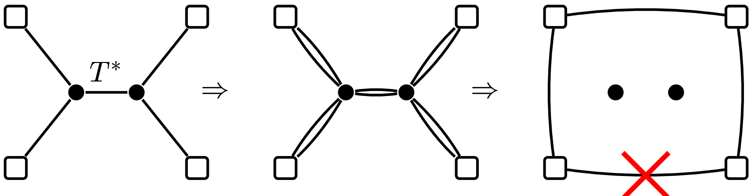
Proof of approximation guarantee

- Claim: $\exists$ spanning tree of cost $\leq 2 \cdot OPT$
- Let T^* be optimum Steiner tree
- Double the edges of T^*
- Observe: Degrees now even $\Rightarrow \exists$ Euler tour $\mathcal{E}$ visiting each terminal

Theorem (Euler)

Given an undirected, connected graph $G = (V, E)$. Then G has an Euler tour (tour containing each edge exactly once) if and only if $|\delta(v)|$ is even for all $v \in V$.

- Shortcut $\mathcal{E}$ such that each terminal is visited once
- Remove an edge $\Rightarrow$ spanning tree of cost $\leq 2 \cdot c(T^*)$ □



State of the art

Known results:

- There is a 1.39-approximation.
- For quasi-bipartite graphs (no Steiner nodes incident): 1.22-apx
- No $< \frac{96}{95}$-apx unless $\mathbf{NP} = \mathbf{P}$.

Part 3
k-Center

Source: *Approximation Algorithms* (Vazirani, Springer Press)

k-Center

Problem: k-CENTER

- Given: Undirected, metric graph $G = (V, E)$, $k \in \mathbb{N}$. Define

$$\ell(v, F) := \min_{u \in F} c_{uv}$$

- Find: k many centers $F \subseteq V$ that minimize the maximum distance from any $v \in V$ to the nearest center:

$$OPT = \min_{F \subseteq V, |F| = k} \max_{v \in V} \{\ell(v, F)\}$$

The algorithm

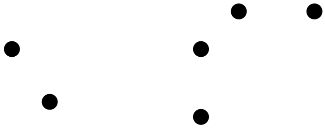
Algorithm:

(1) Guess $OPT \in \{c_{uv} \mid u, v \in V\}$

(2) $F := \emptyset$

(3) REPEAT

  (4) IF $\exists v \in V : \ell(v, F) > 2 \cdot OPT$ THEN $F := F \cup \{v\}$
  ELSE RETURN F



The algorithm

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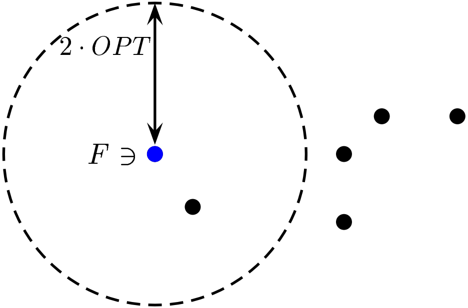

The algorithm

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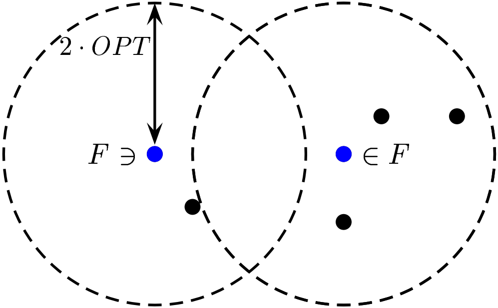


Guessing

For simplicity we sometimes **guess** parameters:

Algorithm with guessing:

(1) Guess a parameter m

(2) $\ldots$ compute a solution $\mathcal{S}$ using m $\ldots$

(3) return $\mathcal{S}$

Algorithm without guessing:

(1) FOR all choices of m DO

  (2) $\ldots$ compute a solution $\mathcal{S}(m)$ $\ldots$

(3) return the best found solution $\mathcal{S}(m)$

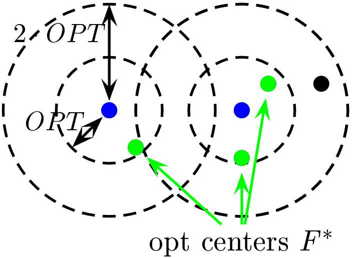
- Still polynomial if the domain of m is polynomial
- Typical guesses: OPT, $O(1)$ many nodes in a graph

The analysis

Theorem

One has $|F| \leq k$ and $\ell(v, F) \leq 2 \cdot OPT$ for all $v \in V$.

- $\ell(v, F) \leq 2 \cdot OPT$, otherwise algo would not have stopped.
- Remains to show $|F| \leq k$.
- Let $F^* \subseteq V, |F^*| = k$ be optimum solution.
- Observe: $c_{uv} > 2 \cdot OPT \; \forall u, v \in F : u \neq v$
- Hence the centers in F^* that serve u and v must be different $\Rightarrow |F| \leq |F^*| \leq k$.

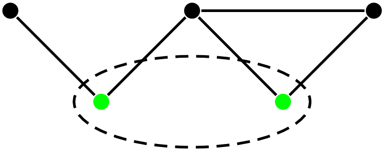


Dominating Set

Problem: DOMINATING SET

- Given: Undirected graph $G = (V, E)$
- Find: Dominating set $U \subseteq V$ of minimum size

$$OPT_{DS} = \min\{|U| \mid U \subseteq V, U \cup \bigcup_{u \in U} \delta(u) = V\}$$



dominating set

Theorem

Given (G, k), it is **NP***-hard to decide, whether $OPT_{DS} \leq k$.*

Hardness of k-Center

Theorem

Unless $\mathbf{NP} = \mathbf{P}$*, for all* $\varepsilon > 0$*, there is no* $(2-\varepsilon)$*-approximation algorithm for* k-CENTER*.*

- Let (G, k) be DOMINATINGSET instance.
- Suppose A is a $(2-\varepsilon)$-algorithm for k-Center
- Define complete graph G' on nodes V with

$$c(u,v) := \begin{cases} 1 & (u,v) \in E \\ 2 & \text{otherwise} \end{cases}$$

- $\exists$ DS of size $\leq k \Rightarrow k$-Center solution with value 1
- $\exists k$-CENTER solution with value $\leq 1 \Rightarrow \exists$ DS of size $\leq k$
- Run A on G':
  - $A(G') < 2 \Rightarrow A(G') = 1 \Rightarrow$ answer to DS instance is YES
  - $A(G') \geq 2 \Rightarrow$ answer is NO □

Part 4
Traveling Salesman Problem

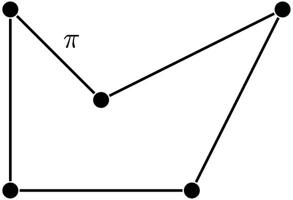
Source: *Approximation Algorithms* (Vazirani, Springer Press)

TSP

Problem: TRAVELING SALESMAN PROBLEM (TSP)

- Given: Undirected graph $G = (V, E)$ with metric cost $c : E \to \mathbb{Q}_+$
- Find: Minimum cost tour visiting all nodes

$$\min_{\text{tour } \pi: V \to V} \left\{ \sum_{v \in V} c(v, \pi(v)) \right\}$$



A 2-approximation for TSP

Algorithm:

(1) Compute an MST T on G

(2) Double the edges in T

(3) Compute Euler tour $\mathcal{E}$ using edges in T

(4) Shortcut to obtain a tour π

Theorem

Algorithm yields a 2-apx.

- Let π^* be optimum tour
- $\exists$ a spanning tree on G of cost $c(T) \leq OPT$ (just delete an arbitrary edge from π^*)
- Degrees are even after doubling, hence $\mathcal{E}$ exists and $c(\mathcal{E}) \leq 2 \cdot OPT$
- $c(\pi) \leq 2 \cdot OPT$ (G is metric, hence shortcutting does not increase the cost) $\square$

A 3/2-approximation for TSP

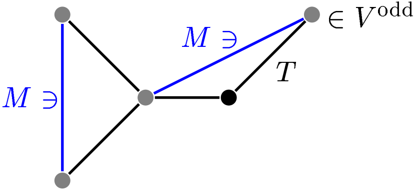
Algorithm (Christofides):

(1) Compute an MST T

(2) Find min cost perfect matching M on nodes $V^{\text{odd}} \subseteq V$ with odd degree in T

(3) Find Euler tour in $T \cup M$.

(4) Return π obtained by shortcutting the Euler tour



Reminder

A perfect matching in an undirected graph $G' = (V', E')$ is an edge set $M \subseteq E'$ with $|\delta_M(v)| = 1 \; \forall v \in V'$. The cheapest perfect matching can be found in poly-time.

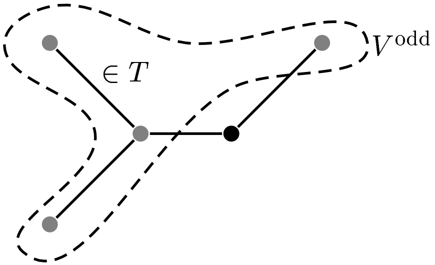
A 3/2-approximation for TSP (2)

Theorem

The algorithm gives a 3/2-apx.

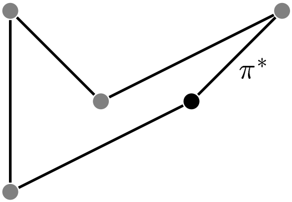
- Again $c(T) \leq OPT$
- $V^{\text{odd}} := \{v \in V \mid |\delta_T(v)| \text{ odd}\}$.
- Claim: $|V^{\text{odd}}|$ is even because

$$|V^{\text{odd}}| \equiv_2 \sum_{v \in V^{\text{odd}}} |\delta_T(v)| \equiv_2 \sum_{v \in V} |\delta_T(v)| \equiv_2 0$$



A 3/2-approximation for TSP (3)

- Let π^* be optimum tour. Obtain shortcutted tour π^{odd} on V^{odd}: $c(\pi^{\text{odd}}) \leq OPT$.
- Partition π^{odd} into 2 matchings M_1, M_2 on V^{odd}
- Let $M \in \{M_1, M_2\}$ be the cheaper of both matchings
- $c(M) \leq \frac{1}{2}c(\pi^{\text{odd}}) \leq \frac{1}{2}OPT$
- In $T \cup M$ all nodes have even degree, hence $T \cup M$ contains an Euler tour of cost $\leq c(T) + c(M) \leq \frac{3}{2}OPT$.



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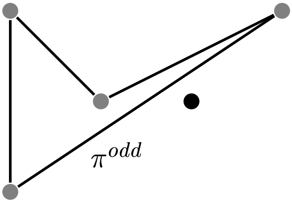

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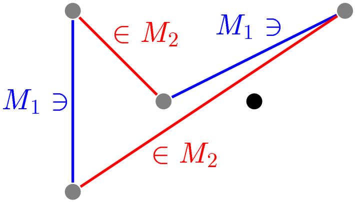
- In $T \cup M$ all nodes have even degree, hence $T \cup M$ contains an Euler tour of cost $\leq c(T) + c(M) \leq \frac{3}{2} OPT$.



Open Problems on TSP

Open Problem

- Is there a $< 3/2$-apx for TSP?
- Held-Karp LP relaxation is conjectured to have integrality gap $4/3$.
- No $(\frac{5381}{5380} - \varepsilon)$-apx even if $c_e \in \{1, 2\}$

PART 5
THE CAPACITATED VEHICLE ROUTING PROBLEM

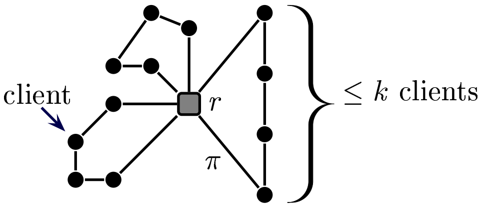
SOURCE: *Bounds and Heuristics for capacitated routing problems* (Haimovich, Rinnooy Kan)

http://www.jstor.org/stable/3689422

The Capacitated Vehicle Routing Problem

Problem: CVRP

- Given: Undirected graph $G = (C \cup \{r\}, E)$ with metric costs $c : E \to \mathbb{Q}_+$, depot r, clients C and vehicle capacity k
- Find: A tour π of minimal cost which visits all clients at least once, but must revisit the depot after each $\leq k$ client visits



Assume: $|C| = \mathbb{Z} \cdot k$ (otherwise add clients at the depot)

A 5/2-apx for CVRP

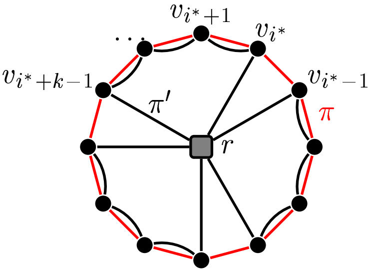
Algorithm:

(1) Compute a 3/2-approximate TSP tour π on clients

(2) Let $v_0, \ldots, v_{n-1}$ be clients in visiting order

(3) Choose randomly a starting node v_{i^*}

(4) Starting from v_{i^*} revisit r every k many clients (i.e. augment the tour with edges $r \to v_i, v_{i-1} \to r$ if $i \equiv_k i^*$) to obtain a CVRP solution π'

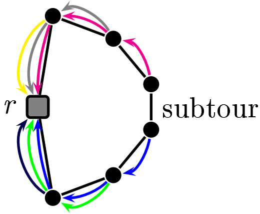


The analysis

Lemma

$E[APX] \leq \frac{5}{2}OPT$

- Opt. TSP tour costs $OPT_{\text{TSP}} \leq OPT$ hence $c(\pi) \leq \frac{3}{2}OPT$
- $\Pr[\text{need edge } (r, v_i)] = \frac{2}{k}$
- $E[APX] \leq c(\pi) + \frac{2}{k}\sum_{v \in C} c(r, v)$
- Look at a subtour in optimum CVRP solution. Send $k/2$ clients [counter-]clockwise to r: edges in subtour used $\leq k/2$ times $\Rightarrow \sum_{v \in C} c(v, r) \leq \frac{k}{2}OPT$



$$E[APX] \leq c(\pi) + \frac{2}{k}\sum_{v \in C} c(r, v) \leq \frac{3}{2}OPT + \frac{2}{k} \cdot \frac{k}{2}OPT = \frac{5}{2}OPT$$

Part 6
Set Cover

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Set Cover

Problem: SET COVER

- Given: Elements $U := \{1, \ldots, n\}$, sets $S_1, \ldots, S_m \subseteq U$ with cost $c(S_i)$
- Find:

$$OPT = \min_{I \subseteq \{1,\ldots,m\}} \left\{ \sum_{i \in I} c(S_i) \mid \bigcup_{i \in I} S_i = U \right\}$$

Greedy algorithm:

(1) $I := \emptyset$
(2) WHILE not yet all elements covered DO
   (3) $price(S) := \dfrac{c(S)}{|S \setminus \bigcup_{i \in I} S_i|}$
   (4) $I := I \cup \{$ set S with minimum $price(S)\}$

Theorem

The greedy algorithm yields a $O(\log n)$-approximation.

Analysis

- Let $e_1, \dots, e_n$ be elements in the order of covering.
- Suppose S ($S \in I$) newly covered $e_k, \dots, e_\ell$

$$e_1, e_2, e_3, \dots, \overbrace{\underbrace{e_k, \dots, e_j, \dots, e_\ell}_{\text{covered by } S}, \dots, e_n}^{n-k+1 \text{ elements}}$$

- Define $price(e_j) := price(S)$ for $j \in \{k, \dots, \ell\}$.
- Consider the iteration, when S was chosen: Still $n - k + 1$ elements where uncovered and it was still possible to cover them all at cost OPT. Since S minimizes the price:

$$price(e_j) = price(e_k) \leq \frac{OPT}{n-k+1} \leq \frac{OPT}{n-j+1}$$

- Finally

$$APX = \sum_{j=1}^{n} price(e_j) \leq \sum_{j=1}^{n} \frac{OPT}{n-j+1} = OPT \cdot \sum_{j=1}^{n} \frac{1}{j} = O(\log n) \cdot OPT$$

Part 7
Set Cover via LPs

Source: *Approximation Algorithms* (Vazirani, Springer Press)

A linear program for SetCover

Introduce decision variables

$$x_i = \begin{cases} 1 & \text{take set } S_i \\ 0 & \text{otherwise} \end{cases}$$

Formulate SetCover as integer linear program:

$$\begin{aligned} \min \sum_{i=1}^{m} c(S_i)x_i & & & \qquad (ILP) \\ \sum_{i:j\in S_i} x_i &\geq 1 & & \forall j \in U \\ x_i &\in \{0,1\} & & \forall i \end{aligned}$$

- Cheapest Set Cover solution = best (ILP) solution

The LP relaxation

We relax this to a linear program

$$
\begin{aligned}
\min \sum_{i=1}^{m} c(S_i) x_i & & & \qquad (LP) \\
\sum_{i : j \in S_i} x_i & \geq & 1 & \quad \forall j \in U \\
0 \leq x_i & \leq & 1 & \quad \forall i
\end{aligned}
$$

- (LP) can be solved in polynomial time (see next chapter)
- Let OPT_f be value of optimum solution
- Of course $OPT_f \leq OPT$
- Integrality gap

$$
\alpha(n) := \sup_{\text{instances } |\mathcal{I}|=n} \frac{OPT(\mathcal{I})}{OPT_f(\mathcal{I})}
$$

The algorithm

Algorithm:

(1) Solve $(LP) \to x^*$ opt. fractional solution

(2) (*Randomized rounding:*) FOR $i = 1, \ldots, m$ DO

  (3) Pick S_i with probability $\min\{\ln(n) \cdot x_i^*, 1\}$

(4) (*Repairing:*) FOR every not covered element $j \in U$ pick the cheapest set containing j

Analysis

Theorem

$E[APX] \leq (\ln(n) + 1) \cdot OPT_f$

Consider an element $j \in U$:

$$\begin{aligned}
\Pr[j \text{ not covered in } (2)] &= \prod_{i: j \in S_i} \Pr[S_i \text{ not picked in } (2)] \\
&\leq \prod_{i: j \in S_i} (1 - \ln(n) \cdot x_i^*) \\
&\overset{1+y \leq e^y}{\leq} \prod_{i: j \in S_i} e^{-\ln(n) \cdot x_i^*} \\
&= e^{-\ln(n) \cdot \overbrace{\sum_{i: j \in S_i} x_i^*}^{\geq 1 \text{ due to LP ineq.}}} \\
&\leq e^{-\ln(n)} = \frac{1}{n}
\end{aligned}$$

Analysis (2)

- Cost of randomized rounding:

$$E[\text{cost in (2)}] = \sum_{i=1}^{m} \Pr[S_i \text{ picked in (2)}] \cdot c(S_i) \leq \sum_{i=1}^{m} \ln(n) x_i^* c(S_i) = \ln(n) \cdot OPT_f$$

- Cost of repairing step: In step (3), we pick n times with prob. $\leq \frac{1}{n}$ a set of cost $\leq OPT_f$. Hence

$$E[\text{cost of step (3)}] \leq n \cdot \frac{1}{n} \cdot OPT_f = OPT_f$$

- By linearity of expectation

$$E[APX] = E[\text{cost in (2)}] + E[\text{cost in (3)}] \leq (\ln(n)+1) \cdot OPT_f \qquad \square$$

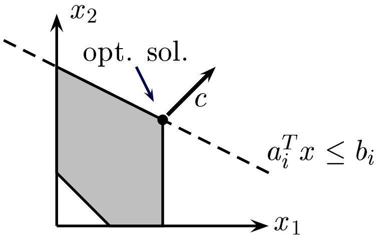
Part 8
Insertion: Linear Programming

Source: *Geometric Algorithms and Combinatorial Optimization* (Grötschel, Lovász, Schrijver)

Linear programs

Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ then

$$\begin{aligned} \max c^T x & & \\ Ax &\leq b \\ x_i &\geq 0 \; \forall i \end{aligned}$$



is called a linear program. Alternatively one might have

- min instead of max
- no non-negativity $x_i \geq 0$
- $Ax = b$

More terminology

- $\text{conv}(\{x,y\}) := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$
- Set $Q \subseteq \mathbb{R}^n$ convex if $\forall x,y \in Q: \text{conv}(\{x,y\}) \subseteq Q$
- A set P is called a polyhedron if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- If P bounded ($\exists M : P \subseteq [-M,M]^n$) then P is a polytope.

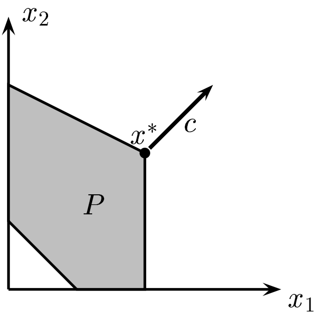
Vertices

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron.

Definition

A point $x^* \in P$ is called a vertex if there is a $c \in \mathbb{R}^n$ such that x^* is the unique optimum solution of $\max\{c^T x \mid x \in P\}$.

Alternative names: basic solution, extreme point.

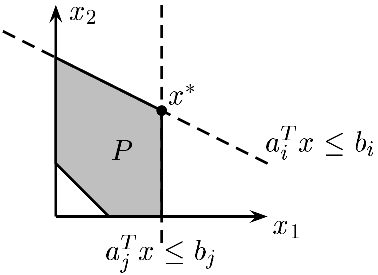


Alternative characterisations

Lemma

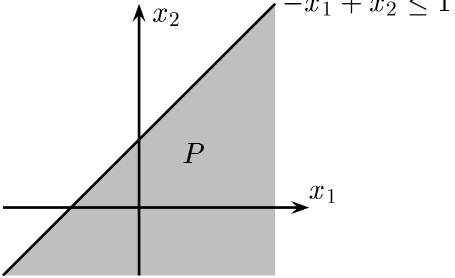
Let $x^* \in P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The following statements are equivalent

- *x^* is a vertex*
- *There are no $y, z \in P$ with (x^*, y, z pairwise different) and $x^* \in conv\{y, z\}$*
- *There is a linear independent subsystem $A'x \leq b'$ (with n constraints) of $Ax \leq b$ s.t. $\{x^*\} = \{x \in \mathbb{R}^n \mid A'x = b'\}$.*



Not every polyhedron has vertices

Example: The polyhedron $P = \{x \in \mathbb{R}^2 \mid -x_1 + x_2 \leq 1\}$ does not have any vertices.



Lemma

Any polytope has vertices.

Lemma

Any polyhedron $P \subseteq \mathbb{R}^n$ with non-negativity constraints $x_i \geq 0\ \forall i = 1, \ldots, n$ has vertices.

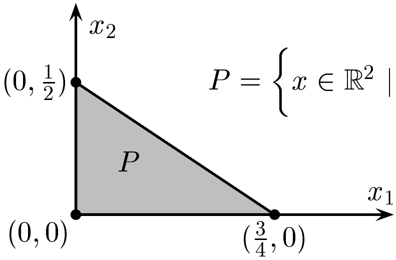
Support of vertex solutions

Lemma

Let x^* be a vertex of

$$P = \{x \in \mathbb{R}^n \mid a_j^T x \leq b_j \; \forall j = 1, \ldots, m; x_i \geq 0 \; \forall i\}$$

Then $|\{i \mid x_i^* > 0\}| \leq m$ *(#non-zero entries* $\leq$ *#constraints).*



$$P = \left\{ x \in \mathbb{R}^2 \mid \begin{array}{l} 4x_1 + 6x_2 \leq 3 \\ x_1 \geq 0; x_2 \geq 0 \end{array} \right\}$$

Proof: There is a subsystem I, J with $|J| + |I| = n$ and $\{x^*\} = \{x \mid a_j^T x = b_j \; \forall j \in J; \; x_i = 0 \; \forall i \in I\}$. Hence $|I| = n - |J| \geq n - m$.

Linear programming is doable in polytime

Theorem

Given $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$, there is an algorithm which solves

$$\max\{c^T x \mid Ax \leq b\}$$

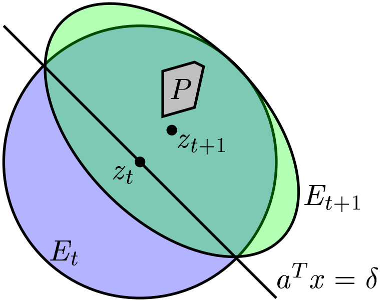
in time polynomial in n, m and the encoding length of A, b, c. The algorithm returns an optimum vertex solution if there is any.

- Polynomial here means that the number of bit operations is bounded by a polynomial (Turing model).
- Encoding length (= #bits used to encode an object) for
  - integer $\alpha \in \mathbb{Z}$: $\langle \alpha \rangle := \lceil \log_2(|\alpha| + 1) \rceil + 1$.
  - rational number $\alpha = \frac{p}{q} \in \mathbb{Q}$: $\langle \alpha \rangle := \langle p \rangle + \langle q \rangle$
  - vector $c \in \mathbb{Q}^n$: $\langle c \rangle := \sum_{i=1}^n \langle c_i \rangle$
  - inequality $a^T x \leq \delta$: $\langle a \rangle + \langle \delta \rangle$
  - matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$: $\langle A \rangle := \sum_{i=1}^m \sum_{j=1}^n \langle a_{ij} \rangle$

The ellipsoid method

Input: Fulldimensional polytope $P \subseteq \mathbb{R}^n$
Output: Point in P

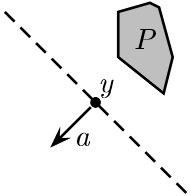
(1) Find ellipsoid $E_1 \supseteq P$ with center z_1
(2) FOR $t = 1, ..., \infty$ DO
    (3) IF $z_t \in P$ THEN RETURN z_t
    (4) Find hyperplane $a^T x = \delta$ through z_t such that $P \subseteq \{x \mid a^T x < \delta\}$
    (5) Compute ellipsoid $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq \delta\}$ with $\mathrm{vol}(E_{t+1}) = (1 - \frac{\Theta(1)}{n})\mathrm{vol}(E_t)$



The ellipsoid method (2)

Problem: SEPARATION PROBLEM FOR P:

- Given: $y \in \mathbb{Q}^n$
- Find: $a \in \mathbb{Q}^n$ with $a^T y > a^T x \; \forall x \in P$ (or assert $y \in P$).



Rule of thumb

If one can solve the SEPARATION PROBLEM for $P \subseteq \mathbb{R}^n$ in poly-time, then one can solve $\max\{c^T x \mid x \in P\}$ efficiently.

Important: The number of inequalities does not play a role. Especially we can optimize in many cases even if the number of inequalities is exponential.

Theorem

Let $P \subseteq \mathbb{R}^n$ be a polyhedron that can be described as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$, and let $c \in \mathbb{Q}^n$ be an objective function. Let φ be an upper bound on

- *the encoding length of each <u>single</u> inequality in $Ax \leq b$.*
- *the dimension n*
- *the encoding length of c.*

Suppose one can solve the following problem in time poly(φ):

> **Separation problem:** *Given $y \in \mathbb{Q}^n$ with encoding length poly(φ) as input. Decide, whether $y \in P$. If not find an $a \in \mathbb{Q}^n$ with $a^T y > a^T x \; \forall x \in P$.*

Then there is an algorithm that yields in time poly(φ) either

- *$x^* \in \mathbb{Q}^n$ attaining $\max\{c^T x \mid x \in P\}$ (x^* will be a vertex if P has vertices)*
- *P empty*
- *Vectors $x, y \in \mathbb{Q}^n$ with $x + \lambda y \in P \; \forall \lambda \geq 0$ and $c^T y \geq 1$.*

Here running times are w.r.t. the Turing machine model.

Weak duality

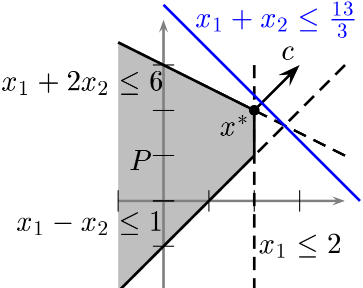
Observation

Consider the LP $\max\{c^T x \mid x \in P\}$ with $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Let $y \geq \mathbf{0}$. Then $(y^T A)x \leq y^T b$ is a feasible inequality for P (i.e. $(y^T A)x \leq y^T b \; \forall x \in P$). In fact, if $y^T A = c^T$, then

$$c^T x = (y^T A)x \leq y^T b \quad \forall x \in P$$

Example: $\max\{x_1 + x_2 \mid x_1 + 2x_2 \leq 6, \; x_1 \leq 2, \; x_1 - x_2 \leq 1\}$
Optimum solution: $x^* = (2, 2)$ with $c^T x^* = 4$.

$$\begin{array}{llllll}
\frac{2}{3} \cdot (& x_1 & +2x_2 & \leq & 6) & \\
0 \cdot (& x_1 & & \leq & 2) & \\
\frac{1}{3} \cdot (& x_1 & -x_2 & \leq & 1) & \\
\hline
 & x_1 & +x_2 & \leq & \frac{13}{3} & \approx 4.33
\end{array}$$



Weak duality (2)

Theorem (Weak duality)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. *Then*

$$\underbrace{\max\{c^T x \mid Ax \leq b\}}_{(P)} \leq \underbrace{\min\{b^T y \mid y^T A = c^T;\ y \geq \mathbf{0}\}}_{(D)}$$

given that both systems are feasible.

- If (P) is the primal program, then (D) is the dual program to (P).
- Note: The dual of the dual is the primal.

Strong duality

Theorem (Strong duality I)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. *Then*

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid y^T A = c^T;\ y \geq \mathbf{0}\}$$

given that both systems are feasible.

Theorem (Strong duality II)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. *Then*

$$\max\{c^T x \mid Ax \leq b, x \geq \mathbf{0}\} = \min\{b^T y \mid y^T A \geq c^T, y \geq \mathbf{0}\}$$

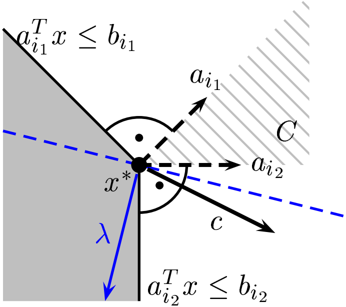
given that both systems are feasible.

Hand-waving proof of strong duality

Claim

Let x^* be optimum solution of $\max\{c^T x \mid Ax \leq b\}$. Then there is a $y \geq \mathbf{0}$ with $y^T A = c^T$ and $y^T b = c^T x^*$.

- Let $a_1, \ldots, a_m$ be rows of A.
- Let $I := \{i \mid a_i^T x^* = b_i\}$ be the tight inequalities.



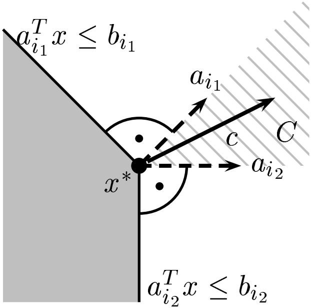
- Suppose for contradiction $c \notin \{\sum_i a_i y_i \mid y_i \geq 0, i \in I\} =: C$
- Then there is a $\lambda \in \mathbb{R}^n$ with $c^T \lambda > 0$, $a_i^T \lambda \leq 0 \; \forall i \in I$.
- Walking in direction λ improves objective function. But x^* was optimal. Contradiction!

Hand-waving proof of strong duality

Claim

Let x^* be optimum solution of $\max\{c^T x \mid Ax \leq b\}$. Then there is a $y \geq \mathbf{0}$ with $y^T A = c^T$ and $y^T b = c^T x^*$.

- Let $a_1, \ldots, a_m$ be rows of A.
- Let $I := \{i \mid a_i^T x^* = b_i\}$ be the tight inequalities.



- $\exists y \geq \mathbf{0} : y^T A = c^T$ and $y_i = 0 \; \forall i \notin I$ (we only use tight inequalities)

$$y^T b - c^T x^* = y^T b - y^T A x^* = y^T (b - Ax^*) = \sum_{i=1}^{m} \underbrace{y_i}_{=0 \text{ if } i \notin I} \cdot \underbrace{(b_i - a_i^T x^*)}_{=0 \text{ if } i \in I} = 0$$

Complementary Slackness

Warning: Primal and dual are switched here.

Theorem (Complementary slackness)

Let x^* be a solution for

$$(P) : \min\{c^T x \mid Ax \geq b, x \geq \mathbf{0}\}$$

and y^* a solution for

$$(D) : \max\{b^T y \mid A^T y \leq c, y \geq \mathbf{0}\}.$$

Let a_i be the ith row of A and a^j be its jth column. Then x^* and y^* are both optimal $\Leftrightarrow$ both following conditions are true

- *Primal complementary slackness:* $x_j > 0 \Rightarrow (a^j)^T y = c_j$
- *Dual complementary slackness:* $y_i > 0 \Rightarrow a_i^T x = b_i$

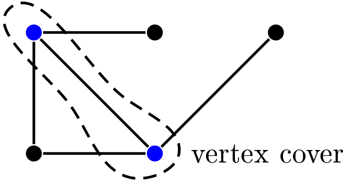
Part 9
Weighted Vertex Cover

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Vertex Cover

Problem: Weighted Vertex Cover

- Given: Undirected graph $G = (V, E)$, node weights $c : V \to \mathbb{Q}_+$
- Find: Subset $U \subseteq V$ such that every edge is incident to at least one node in U and $\sum_{v \in U} c(v)$ is minimized.



Consider the LP

$$\min \sum_{v \in V} c(v) x_v$$

$$\begin{aligned} x_u + x_v &\geq 1 \quad \forall (u,v) \in E \\ x_v &\geq 0 \quad \forall v \in V \end{aligned}$$

Half-integrality

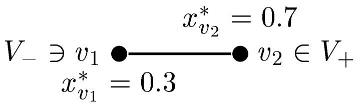
Lemma

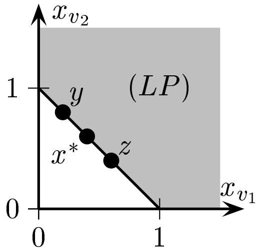
Let x^* be a basic solution of (LP). Then $x^*_v \in \{0, \frac{1}{2}, 1\}$ for all $v \in V$, i.e. x^* is <u>*half-integral*</u>.

- Suppose x^* is not half-integral, i.e. not both sets are empty:

$$V_+ := \left\{v \mid \frac{1}{2} < x^*_v < 1\right\}, V_- := \left\{v \mid 0 < x^*_v < \frac{1}{2}\right\}$$

- It suffices to show that x^* can be written as convex combination $x^* = \frac{1}{2}y + \frac{1}{2}z$ for 2 different feasible (LP) solutions y, z.

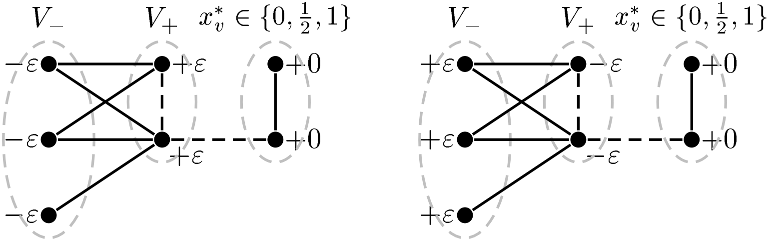




Half-integrality (2)

- Define

$$y_v := \begin{cases} x_v^* + \varepsilon & x_v^* \in V_+ \\ x_v^* - \varepsilon & x_v^* \in V_- \\ x_v^* & \text{otherwise} \end{cases} \quad \text{and} \quad z_v := \begin{cases} x_v^* - \varepsilon & x_v^* \in V_+ \\ x_v^* + \varepsilon & x_v^* \in V_- \\ x_v^* & \text{otherwise} \end{cases}$$



- Tight edges $(u, v) \in E : x_v^* + x_u^* = 1$ drawn solid
- Constraints satisfied by y, z for $\varepsilon > 0$ small enough. □

The Algorithm

Algorithm:

(1) Compute an optimum basic solution x^* to (LP)

(2) Choose vertex cover $U := \{v \mid x^*_v > 0\}$

Theorem

U is a vertex cover of cost $\leq 2 \cdot OPT_f$.

Proof.

Clearly U is feasible. Furthermore

$$\sum_{v \in U} c(v) = \sum_{v \in V} \lceil x^*_v \rceil c(v) \leq 2 \sum_{v \in V} x^*_v c(v) = 2 \cdot OPT_f.$$

□

Inapproximability

Theorem (Khot & Regev '03)

There is no polynomial time $(2 - \varepsilon)$-apx unless Unique Games Conjecture is false.

Unique Games Conjecture

For all $\varepsilon > 0$, there is a prime $p := p(\varepsilon)$ such that the following problem is **NP**-hard:

- Given: Equations $x_i \equiv_p a_{ij} x_j$ for some (i, j) pairs
- Distinguish:
  - Yes: max satisfiable fraction $\geq 1 - \varepsilon$
  - No: max satisfiable fraction $\leq \varepsilon$

Example:

$$\begin{aligned} x_1 &\equiv_{13} 4 \cdot x_3 \\ x_2 &\equiv_{13} 9 \cdot x_1 \\ &\cdots \end{aligned}$$

Part 10
Insertion: Algorithmic probability theory

Source: *Probability and Computing* (Mitzenmacher & Upfal, Cambridge Press)

Probability theory

Definition

A (discrete) probability space consists of

- A (countable) sample space Ω modelling all possible outcomes of a random process.
- A probability function $\Pr : 2^{\Omega} \to \mathbb{R}$ such that
  - (a) $0 \leq \Pr[E] \leq 1 \; \forall E \subseteq \Omega$
  - (b) $\Pr[\Omega] = 1$
  - (c) For any (countable) sequence of pairwise disjoint events $E_1, E_2, \ldots \subseteq \Omega$

$$\Pr\left[\bigcup_{i \geq 1} E_i\right] = \sum_{i \geq 1} \Pr[E_i]$$

Definition (Random variable)

A function $X : \Omega \to \mathbb{R}$ is called a random variable.

Probability theory (2)

Definition (Expectation)

Let $X : \Omega \to \mathbb{R}$ be a random variable. Then

$$E[X] = \sum_i i \cdot \Pr[X = i]$$

Lemma (Linearity of expectation)

Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ random variables with finite expectations. Then

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Probability theory (3)

Lemma (Independence)

Random variables $X_1, \ldots, X_n$ are called independent if

$$\forall I \subseteq \{1, \ldots, n\} : \forall x_i : \Pr\left[\bigcap_{i \in I} (X_i = x_i)\right] = \prod_{i \in I} \Pr[X_i = x_i]$$

Lemma

Let $X_1, \ldots, X_n$ <u>independent</u> random variables. Then

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

Probability theory (4)

Lemma (Union bound)

Let $E_1, \ldots, E_n \subseteq \Omega$ *be events*

$$\Pr\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \Pr[E_i]$$

Probability theory (5)

Lemma (Markov bound)

Let $X \geq 0$ *be a random variable. Then*

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$

Proof.

The value $E[X]$ is

$$\begin{aligned} E[X] &= \underbrace{E[X \mid X \geq a]}_{\geq a} \cdot \Pr[X \geq a] + \underbrace{E[X \mid X < a]}_{\geq 0} \cdot \underbrace{\Pr[X < a]}_{\geq 0} \\ &\geq a \cdot \Pr[X \geq a] \end{aligned}$$

$\square$

Probability theory (6)

Theorem (Chernov bound)

Let $X_1, \ldots, X_n$ *be independent random variables with* $X_i \in \{0,1\}$ *and* $X := X_1 + \ldots + X_n$*. For any* $\delta > 0$ *one has*

$$\Pr[X \geq (1+\delta)E[X]] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{E[X]}$$

Let $t := \ln(1+\delta) > 0$, $p_i := \Pr[X_i = 1]$. Note that $E[X_i] = p_i$.

$$
\begin{aligned}
\Pr[X \geq (1+\delta)E[X]] &\overset{e^{tx}\text{ mon.inc.}}{=} \Pr[e^{tX} \geq e^{t(1+\delta)E[X]}] \\
&\overset{\text{Markov}}{\leq} \frac{E[e^{tX}]}{e^{t(1+\delta)E[X]}} \\
&\leq \frac{E[\prod_{i=1}^n e^{tX_i}]}{e^{t(1+\delta)E[X]}} \\
&\overset{X_1,\dots,X_n\text{ indep}}{=} \frac{\prod_{i=1}^n E[e^{tX_i}]}{e^{t(1+\delta)E[X]}} \\
&\overset{(*)}{\leq} \frac{\prod_{i=1}^n e^{\delta p_i}}{e^{t(1+\delta)E[X]}} \\
&= \frac{e^{\delta \sum_{i=1}^n p_i}}{e^{t(1+\delta)E[X]}} \\
&\overset{E[X]=\sum_{i=1}^n p_i}{=} \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{E[X]}
\end{aligned}
$$

$$
(*) \quad E[e^{tX_i}] = p_i \cdot \underbrace{e^{t\cdot 1}}_{=1+\delta} + (1-p_i) \cdot \underbrace{e^{t\cdot 0}}_{=1} = 1 + \delta p_i \leq e^{\delta p_i} \qquad \square
$$

Probability theory (7)

Theorem (Variants of Chernov bound)

Let $X_1, \ldots, X_n \in \{0,1\}$ *be independent random variables with and* $X := X_1 + \ldots + X_n$ *and* $0 < \delta \leq 1$. *Then*

- *Let* $\mu \geq E[X]$, *then*

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu \cdot \delta^2/2}$$

- *Let* $\mu \leq E[X]$, *then*

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\mu \cdot \delta^2/2}$$

Part 11
Minimizing Congestion

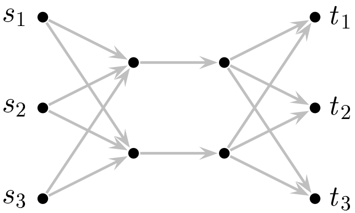
Source: *Randomized rounding: A technique for provably good algorithms and algorithmic proofs* (Raghavan, Tompson)
http://www.springerlink.com/content/n16347864k45367w/fulltext.pdf

Minimizing Congestion

Problem: MinCongestion

- Given: Directed graph $G = (V, E)$ with demand pairs (s_i, t_i) $s_i, t_i \in V$, $i = 1, \ldots, k$
- Find: s_i-t_i paths P_i that minimize the **congestion**

$$\max_{e \in E} |\{i : e \in P_i\}|$$



Minimizing Congestion

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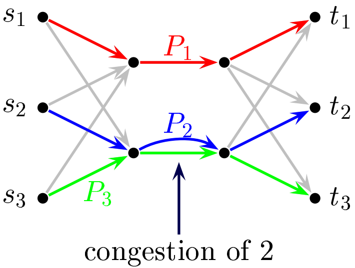


A flow-based LP formulation of MinCongestion

$$\min C \qquad (LP)$$

$$\sum_{e \in \delta^+(v)} f_i(e) - \sum_{e \in \delta^-(v)} f_i(e) = \begin{cases} 1 & v = s_i \\ -1 & v = t_i \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{i=1}^{k} f_i(e) \leq C \quad \forall e \in E$$

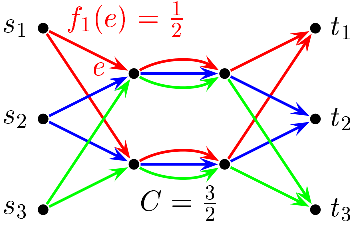
$$C \geq 1$$

$$f_i(e) \geq 0 \quad \forall i \; \forall e \in E$$

$f_1(e) = \frac{1}{2}$ on red e

$f_2(e) = \frac{1}{2}$ on blue e

$f_3(e) = \frac{1}{2}$ on green e



Path Decomposition

- **Input:** s-t flow $f : E \to \mathbb{Q}_+$ (without directed cycles)
- **Output:** Paths $p_1, \dots, p_m$ with values $v_1, \dots, v_m \geq 0$

(1) $i := 1$

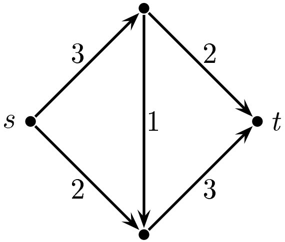
(2) WHILE $f \neq \mathbf{0}$ DO

   (3) Let p_i be *any* s-t path in $\{e \mid f(e) > 0\}$

   (4) $v_i := \min\{f(e) \mid e \in p_i\}$

   (5) $f(e) := f(e) - v_i \; \forall e \in p_i$

   (6) $i := i + 1$



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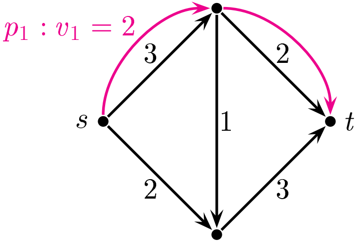
  (4) $v_i := \min\{f(e) \mid e \in p_i\}$

  (5) $f(e) := f(e) - v_i \; \forall e \in p_i$

  (6) $i := i + 1$



Path Decomposition

- **Input:** s-t flow $f : E \to \mathbb{Q}_+$ (without directed cycles)
- **Output:** Paths $p_1, \dots, p_m$ with values $v_1, \dots, v_m \geq 0$

(1) $i := 1$

(2) WHILE $f \neq \mathbf{0}$ DO

  (3) Let p_i be *any* s-t path in $\{e \mid f(e) > 0\}$

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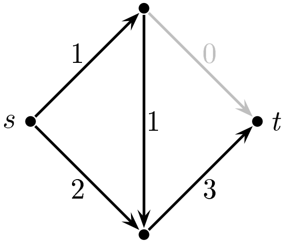

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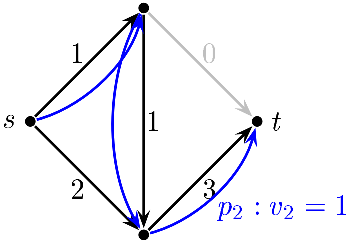

Path Decomposition

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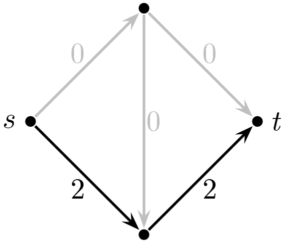

Path Decomposition

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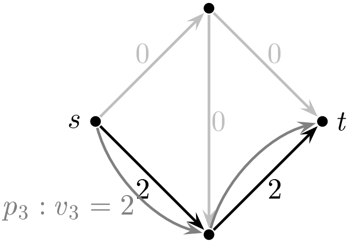

Path Decomposition

- **Input:** s-t flow $f : E \to \mathbb{Q}_+$ (without directed cycles)
- **Output:** Paths $p_1, \ldots, p_m$ with values $v_1, \ldots, v_m \geq 0$

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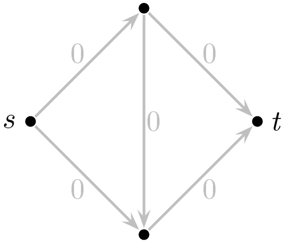

Path Decomposition

- **Input:** s-t flow $f : E \to \mathbb{Q}_+$ (without directed cycles)
- **Output:** Paths $p_1, \ldots, p_m$ with values $v_1, \ldots, v_m \geq 0$

(1) $i := 1$

(2) WHILE $f \neq \mathbf{0}$ DO

  (3) Let p_i be *any* s-t path in $\{e \mid f(e) > 0\}$

  (4) $v_i := \min\{f(e) \mid e \in p_i\}$

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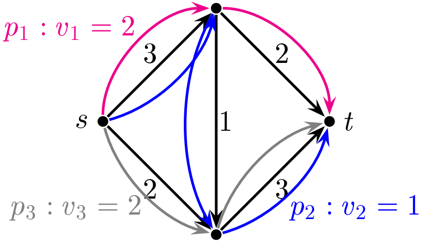


Path Decomposition

Lemma

The algorithm decomposes the flow in s-t paths $p_1, \ldots, p_m$ with $m \leq |E|$.

$$\sum_{e \in \delta^+(s)} f(e) = \sum_{i=1}^{m} v_i \quad \text{and} \quad \sum_{i : e \in p_i} v_i = f(e) \; \forall e \in E$$

- f remains a flow throughout the algorithm.
- In each iteration there is an edge, where the flow drops down to 0.

An approximation algorithm for MinCongestion

Algorithm

(1) Solve $(LP) \to$ flows $f_1, \ldots, f_k$ frac. congestion OPT_f

(2) FOR $i = 1, \ldots, k$ DO

  (3) apply path decomposition to $f_i \to (p^i_j, v^i_j)$ $(\sum_j v^i_j = 1 \; \forall i)$

(4) Choose P_i among p^i_j's with $\Pr[P_i = p^i_j] = v^i_j$

Theorem

With probability $\geq 1 - \frac{1}{n}$ *the congestion is* $\leq O(\frac{\ln n}{\ln \ln n}) \cdot OPT_f$.

- Consider any edge $e \in E$.
- Let $X^e_i \in \{0, 1\}$ be the random variable, saying whether the s_i-t_i path uses e. $X^e_1, \ldots, X^e_k$ are independent!
- Let $X^e := \sum_{i=1}^k X^e_i$ be the number of paths, crossing e.
- $E[X^e] = \sum_{i=1}^k \underbrace{\Pr[X^e_i]}_{=f_i(e)} = \sum_{i=1}^k f_i(e) \leq OPT_f$.

Proof (2)

$$\begin{aligned}
\Pr\Big[X^e > \Big(\overbrace{c\frac{\log n}{\log\log n}}^{=:\delta}+1\Big)\overbrace{OPT_f}^{\geq E[X^e]}\Big] &\leq \left(\frac{e^\delta}{\delta^\delta}\right)^{\overbrace{OPT_f}^{\geq 1}} \\
&\leq \left(\frac{e}{c\frac{\ln n}{\ln\ln n}}\right)^{c\frac{\ln n}{\ln\ln n}} \\
&\overset{c\geq 3}{\leq} \left(\frac{\ln\ln n}{\ln n}\right)^{c\frac{\ln n}{\ln\ln n}} \\
&= \Big(\exp\Big(\ln\ln\ln n - \ln\ln n\Big)\Big)^{c\frac{\ln n}{\ln\ln n}} \\
&\overset{n\text{ big}}{\leq} \exp\Big(-\frac{1}{2}\ln\ln n \cdot \frac{c\ln n}{\ln\ln n}\Big) \\
&= \frac{1}{n^{c/2}}
\end{aligned}$$

$$\Pr\Big[\bigvee_{e\in E}\Big(X^e > 6\frac{\ln n}{\ln\ln n}OPT_f\Big)\Big] \leq |E|\cdot\frac{1}{n^3} \leq \frac{1}{n} \quad \square$$

Inapproximability

Theorem (Andrews & Zhang - JACM'08)

There is no $\log^{1-\varepsilon} n$*-apx unless* $\mathbf{NP} \subseteq \mathbf{ZPTIME}(n^{polylog(n)})$.

Part 12
Knapsack

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Knapsack

Problem: Knapsack

- Given: n objects with weight $w_i \in \mathbb{Q}_+$ and profit $p_i \in \mathbb{Q}_+$, size $G \in \mathbb{Q}_+$
- Find: Subset of objects, maximizing the profit and not exceeding the weight bound:

$$OPT = \max_{I \subseteq \{1,\dots,n\}} \left\{ \sum_{i \in I} p_i \mid \sum_{i \in I} w_i \leq G \right\}$$

A dynamic program for KNAPSACK

Dynamic program:

(1) Assume restricted profits $p_i \in \{0, \ldots, B\}$

(2) Compute table entries

$$
\begin{aligned}
T(i,b) &= \min_{I \subseteq \{1,\ldots,i\}} \Big\{ \sum_{j \in I} w_j \mid \sum_{j \in I} p_j \geq b \Big\} \\
&= \text{minimum weight needed for a subset of the first } i \\
&\quad\ \text{objects to obtain a profit of at least } b
\end{aligned}
$$

using dynamic programming

$$
T(i,b) = \min\Big\{ \underbrace{T(i-1,b)}_{\text{don't take } i}, \underbrace{T(i-1,b-p_i) + w_i}_{\text{take } i} \Big\} \ \forall i \ \forall p = 0, \ldots, B
$$

(3) Reconstruct I leading to $\max\{b \in \mathbb{N}_0 \mid T(n,b) \leq G\}$

Observation

The algorithm finds optimum solutions in time $O(n \cdot B)$.

The FPTAS

Algorithm:

(1) Scale profits s.t. $p_{\max} = n/\varepsilon$

(2) Round $p_i' := \lfloor p_i \rfloor$

(3) Compute and return optimum solution I for weights p_i'

Analysis of FPTAS

Theorem

Let $0 < \varepsilon \leq \frac{1}{2}$. The algo gives a $(1+2\varepsilon)$-apx in time $O(n^2/\varepsilon)$.

- W.l.o.g. $OPT \geq p_{\max} = n/\varepsilon$ (we can delete objects that even alone do not fit into the knapsack)
- Let I^* be optimum solution for original profits. Let OPT' be optimum value for profits p'. Then

$$\begin{aligned} OPT' \geq \sum_{i \in I^*} p'_i = \sum_{i \in I^*} \lfloor p_i \rfloor \quad &\geq \quad \sum_{i \in I^*} p_i - |I^*| \geq OPT - n \\ &\geq \quad (1-\varepsilon) OPT \geq \frac{OPT}{1+2\varepsilon} \end{aligned}$$

- Let I be solution found by dynamic program:

$$\sum_{i \in I} p_i \geq \sum_{i \in I} p'_i = OPT' \geq \frac{OPT}{1+2\varepsilon}$$

- $B = \max\{p'_i\} \leq n/\varepsilon$ hence the running time is $O(n^2/\varepsilon)$

Part 13
Multi Constraint Knapsack

Source: Folklore

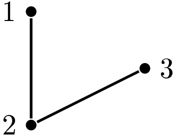
Multi Constraint Knapsack

Problem: Multi Constraint Knapsack (Mck)

- Given: n objects with profits $p_i \in \mathbb{Q}_+$ and k many budgets B_j. Object i has requirement $a_i^j \in \mathbb{Q}_+$ w.r.t. budget j.
- Find: Subset of objects, maximizing the profit and not exceeding any budget:

$$OPT = \max_{I \subseteq \{1,\ldots,n\}} \left\{ \sum_{i\in I} p_i \mid \sum_{i \in I} a_i^j \leq B_j \ \forall j = 1,\ldots,k \right\}$$

- For arbitrary k there is no $n^{1-\varepsilon}$-apx: Take an Independent Set instance $G=(V,E)$. For each edge $e=(u,v)$ add an "edge budget constraint" $a_u^e = a_v^e = 1, B_e = 1$. Then $OPT = OPT_{\text{IS}}$.



$$\Rightarrow \quad \begin{array}{lllll} \max & x_1 & +x_2 & +x_3 & \\ & 1x_1 & +1x_2 & +0x_3 & \leq 1 \\ & 0x_1 & +1x_2 & +1x_3 & \leq 1 \\ & & & x_i & \in \{0,1\} \end{array}$$

A PTAS for $k = O(1)$

Algorithm:

(1) Guess the $\lceil \frac{k}{\varepsilon} \rceil$ items I_{large} in the optimum solution with maximum profit

(2) Let x^* be optimum basic solution to the following LP

$$\begin{aligned}
\max \sum_{i=1}^{n} x_i p_i & & & \\
\sum_{i=1}^{n} a_i^j x_i &\leq& B_j & \quad \forall j = 1, \ldots, k \\
x_i &=& 1 & \quad \forall i \in I_{\text{large}} \\
x_i &=& 0 & \quad \forall i \notin I_{\text{large}} : p_i > \min\{p_j \mid j \in I_{\text{large}}\} \\
0 \leq x_i &\leq& 1 & \quad \forall i = 1, \ldots, n
\end{aligned}$$

(3) Output $I := \{i \mid x_i^* = 1\}$.

The Analysis

Theorem

For constant k the algorithm has polynomial running time. Furthermore $APX \geq (1-\varepsilon)OPT$.

- The produced solution is clearly feasible
- $LP \geq OPT$ (since we guess elements from OPT)
- <u>Observation:</u> $|\{i \mid 0 < x_i^* < 1\}| \leq k$ since x^* is a basic solution and appart from $0 \leq \ldots \leq 1$ there are only k constraints.
- For i with $0 < x_i^* < 1$ one has $p_i \leq \frac{\varepsilon}{k} OPT$

$$\begin{aligned} APX &\geq \sum_{i=1}^{n} \lfloor x_i^* \rfloor p_i \geq LP - \underbrace{\sum_{i:0<x_i^*<1} p_i}_{\leq k \cdot \frac{\varepsilon}{k} OPT} \\ &\geq OPT - k \cdot \frac{\varepsilon}{k} OPT = (1-\varepsilon) OPT \end{aligned}$$

□

Hardness of MultiConstraintKnapsack

Theorem

There is no FPTAS for MultiConstraintKnapsack *even for 2 budgets, unless* $\mathbf{NP} = \mathbf{P}$.

Problem: Partition

- Given: Numbers $a_1, \ldots, a_n \in \mathbb{N}$, $S := \sum_{i=1}^{n} a_i$, $m \in \{1, \ldots, n\}$
- Find: $I \subseteq \{1, \ldots, n\} : |I| = m, \sum_{i \in I} a_i = S/2$

- Recall: Partition is **NP**-hard.
- Define Mck instance with 2 constraints:

$$
\begin{array}{rcll}
\max \sum_{i=1}^{n} x_i & & & \\
\sum_{i=1}^{n} x_i a_i & \leq & S/2 & \\
\sum_{i=1}^{n} x_i (S - a_i) & \leq & S(m - \frac{1}{2}) & \\
x_i & \in & \{0,1\} & \forall i = 1, \ldots, n
\end{array}
$$

Proof

- <u>Claim:</u> $\exists$ PARTITION solution $\Leftrightarrow OPT_{\text{MCK}} \geq m$
- $\Rightarrow$ Suppose $\exists I : |I| = m, \sum_{i\in I} a_i = S/2$. Then this is a MCK solution of value m since

$$\sum_{i\in I}(S - a_i) = mS - \sum_{i\in I} a_i = S(m - \frac{1}{2})$$

- $\Leftarrow$ Let I be MCK solution of value $\geq m$.

$$|I|\cdot S - \frac{S}{2} \overset{\text{1. constr.}}{\leq} |I|\cdot S - \underbrace{\sum_{i\in I} a_i}_{\leq S/2} = \sum_{i\in I}(S - a_i) \overset{\text{2. const.}}{\leq} m\cdot S - \frac{S}{2}$$

- Hence $|I| = m$. Then ineq. holds with "="
- Thus $\sum_{i\in I} a_i = S/2$. $\square$
- Now suppose for contradiction we would have an FPTAS for MCK: Then choose $\varepsilon := \frac{1}{n+1}$. Then the FPTAS would give an optimum solution for the instance resulting from the PARTITION reduction.

Part 14
Bin Packing

Source: *Combinatorial Optimization: Theory and Algorithms* (Korte, Vygen)

Bin Packing

Problem: BinPacking

- Given: Items with sizes $a_1, \dots, a_n \in [0, 1]$
- Find: Assign items to minimum number of bins of size 1.

$$OPT = \min\left\{k \mid \exists I_1 \dot\cup \dots \dot\cup I_k = \{1, \dots, n\} : \forall j : \sum_{i \in I_j} a_i \leq 1\right\}$$

- Define $\text{size}(I) = \sum_{i \in I} a_i$

First Fit

First Fit algorithm:

(1) Start with empty bins

(2) FOR $i = 1, \ldots, n$ DO

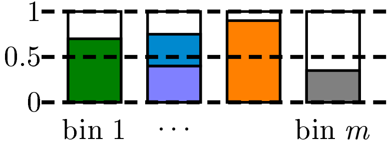
  (3) Assign item i to the bin B with least index such that $a_i + \sum_{j \in B} a_j \leq 1$

Lemma

Let m be the number of used bins. Then $m \leq 2\sum_{i=1}^n a_i + 1 \leq 2 \cdot OPT + 1$.

- All but $m-1$ bins must be filled with $\geq \frac{1}{2}$ (otherwise we would not have opened a new bin):

$$\sum_{i=1}^n a_i \geq \frac{1}{2}(m-1)$$



- Hence $m \leq 2\sum_{i=1}^n a_i + 1$.

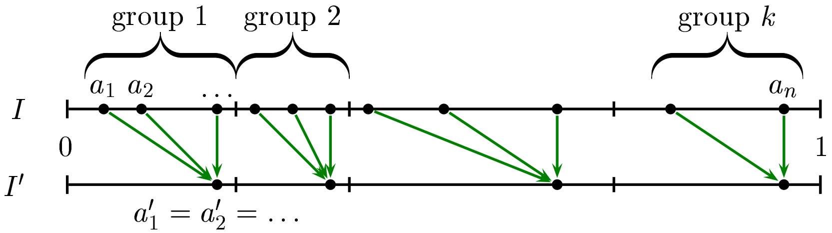
Linear Grouping

- INPUT: Instance $I = (a_1, \ldots, a_n)$, $k \in \mathbb{N}$
- OUTPUT: Instance $I' = (a'_1, \ldots, a'_n)$ with $a'_i \geq a_i$ and $\leq k$ different item sizes

(1) Sort $a_1 \leq a_2 \leq \ldots \leq a_n$

(2) Partition items into k consecutive groups of $\lceil n/k \rceil$ items (the last group might have less items)

(3) Let a'_i be the size of the largest item in i's group

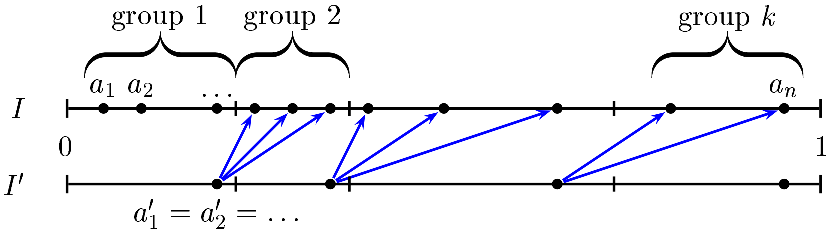


Linear Grouping (2)

Lemma

$OPT(I') \leq OPT(I) + \lceil n/k \rceil$.

- Consider solution $OPT(I)$. Assign item a'_i of group j to a space for item in group $j+1$
- Assign largest $\lceil n/k \rceil$ items to their own bin



An asymptotic PTAS

Algorithm of Fernandez de la Vega & Lueker:

(1) Let $I = \{i \mid a_i > \varepsilon\}$ be set of large items (other items are small)
(2) Apply linear grouping with $k = 1/\varepsilon^2$ groups to $I \to I'$
(3) Compute an optimum distribution of I'
(4) Distribute the small items over the used bins using First Fit

Lemma

The algorithm runs in polynomial time and uses at most $(1 + 2\varepsilon)OPT + 1$ bins.

- Let $b_1, \ldots, b_{1/\varepsilon^2}$ different item sizes in I'.
- Possible bin configurations $\mathcal{P} = \{p \in \{0, \ldots, 1/\varepsilon\}^{1/\varepsilon^2} \mid b^T p \leq 1\}$. $|\mathcal{P}| \leq (1/\varepsilon^2)^{1/\varepsilon}$.
- Solution is described by $(n_p)_{p\in\mathcal{P}}$ (n_p = how many times shall I pack a bin with configuration p?), $n_p \in \{0, \ldots, n\}$
- $\leq n^{(1/\varepsilon^2)^{1/\varepsilon}}$ possibilities for $(n_p)_{p\in\mathcal{P}}$.

An asymptotic PTAS (2)

- We need $OPT(I') + \#$ of bins additionally opened for the small items
- Note that

$$OPT(I') \leq OPT(I) + \lceil |I| \cdot \varepsilon^2 \rceil \leq OPT(I) + \lceil \varepsilon \cdot OPT(I) \rceil = (1+2\varepsilon) \cdot OPT$$

using $OPT(I) \geq \sum_{i \in I} a_i \geq \varepsilon \cdot |I|$ and $OPT \geq OPT(I)$.

- Suppose we need to open an additional bin for small items. Let m be total number of used bins. Then all but one bin are filled to $\geq 1 - \varepsilon$. Hence

$$OPT \geq \sum_{i=1}^{m} a_i \geq (1-\varepsilon) \cdot (m-1)$$

and

$$m \leq \frac{OPT}{1-\varepsilon} + 1 \leq (1+2\varepsilon) OPT + 1$$

Section 14.1
The algorithm of Karmarkar & Karp

The Algorithm of Karmarkar & Karp

Theorem (Karmarkar, Karp '82)

One can compute a BinPacking *solution with* $OPT + O(\log^2 n)$ *many bins in polynomial time.*

- Assume $a_i \geq \delta := \frac{1}{n}$ (again one can distribute items that are smaller than $\frac{1}{n}$ after distributing the large items.

The Gilmore-Gomory LP-relaxation

- Let $b_i \in \mathbb{N}$ now the number of items of size a_i
- n = number of different item sizes
- $m := \sum_{i=1}^{n} b_i$ = total number of items
- $\mathcal{P} = \{p \in \mathbb{Z}_+^n \mid a^T p \leq 1\}$ set of feasible patterns
- Variable x_p = # of bins packed with pattern p

Primal

$$\begin{array}{rcll} \min \mathbf{1}^T x & & & (P(\mathcal{P})) \\ \sum_{p \in \mathcal{P}} x_p p & \geq & b & \\ x & \geq & \mathbf{0} & \end{array}$$

- # var. exponential
- # constr. polynomial

Dual

$$\begin{array}{rcll} \max y^T b & & & (D(\mathcal{P})) \\ p^T y & \leq & 1 & \forall p \in \mathcal{P} \\ y & \geq & \mathbf{0} & \end{array}$$

- # var. polynomial
- # constr. exponential

Idea: Solve the dual with Ellipsoid!

Example

- Item sizes $a_1 = 0.3, a_2 = 0.4$
- # of items $b_1 = 31, b_2 = 7$
- Set of patterns $\mathcal{P} = \left\{ \binom{0}{1}, \binom{0}{2}, \binom{1}{1}, \binom{2}{1}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0} \right\}$

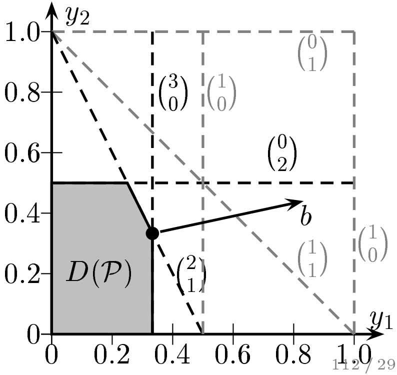
Primal

$$\begin{aligned} \min \mathbf{1}^T x & \\ \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} x &\geq \begin{pmatrix} 31 \\ 7 \end{pmatrix} \\ x &\geq \mathbf{0} \end{aligned}$$

- Opt basic solution is $x = (0, 0, 0, 7, 0, 0, \frac{17}{3})$

Dual

$$\begin{aligned} \max 31y_1 &+ 7y_2 \\ \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 1 \\ 2 & 1 \\ 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix} y &\leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ y &\geq \mathbf{0} \end{aligned}$$



Weak Separation Problem

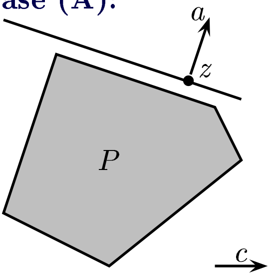
ε-Weak Separation Oracle for $P \subseteq \mathbb{R}^n$, obj.fct. $c \in \mathbb{Q}^n$

INPUT: Vector $z \in \mathbb{Q}^n$

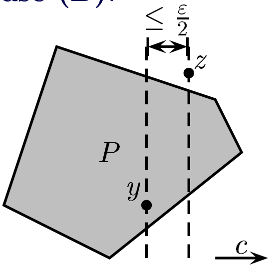
OUTPUT: One of the following

- *Case (A):* Vector a with $a^T x \leq a^T z \; \forall x \in P$
- *Case (B):* Point $y \in P$ with $c^T y \geq c^T z - \frac{\varepsilon}{2}$

Case (A):



Case (B):



- If $z \in P$, just return z ($\rightarrow$ case (B)).

Grötschel-Lovász-Schrijver Algorithm

- INPUT: $c \in \mathbb{Q}^n, x_0 \in \mathbb{Q}^n, \varepsilon, r, R \in \mathbb{Q}_+$: $B(x_0, r) \subseteq P \subseteq B(x_0, R)$
- OUTPUT: $y^* \in P$ with $c^T y^* \geq OPT_f - \varepsilon$

(1) Ellipsod $E_0 := B(x_0, R)$ with center $z_0 := x_0$, $y^* := x_0$

(2) FOR $t = 0, \ldots, poly$ DO

  (4) Submit z_t to ε-weak separation oracle

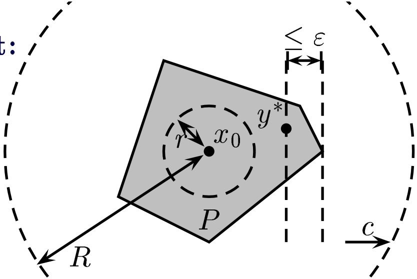
  (5) *Case (A)* $\rightarrow$ *a:* Compute $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq a^T z_t\}$

  (6) *Case (B)* $\rightarrow y \in P$*:*

    (7) IF $c^T y > c^T y^*$ THEN $y^* := y$

    (8) Compute $E_{t+1} \supseteq E_t \cap \{x \mid c^T x \geq c^T z_t\}$

Input/ Output:



Grötschel-Lovász-Schrijver Algorithm

- INPUT: $c \in \mathbb{Q}^n, x_0 \in \mathbb{Q}^n, \varepsilon, r, R \in \mathbb{Q}_+$: $B(x_0, r) \subseteq P \subseteq B(x_0, R)$
- OUTPUT: $y^* \in P$ with $c^T y^* \geq OPT_f - \varepsilon$

(1) Ellipsod $E_0 := B(x_0, R)$ with center $z_0 := x_0$, $y^* := x_0$

(2) FOR $t = 0, \ldots, poly$ DO

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  (5) *Case (A)* $\to a$: Compute $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq a^T z_t\}$

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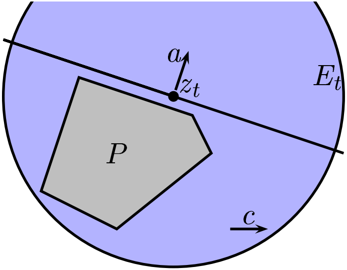
    (8) Compute $E_{t+1} \supseteq E_t \cap \{x \mid c^T x \geq c^T z_t\}$

Case (A):



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- INPUT: $c \in \mathbb{Q}^n, x_0 \in \mathbb{Q}^n, \varepsilon, r, R \in \mathbb{Q}_+$: $B(x_0, r) \subseteq P \subseteq B(x_0, R)$
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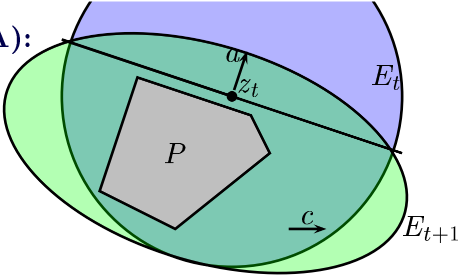
Case (A):



Grötschel-Lovász-Schrijver Algorithm

- INPUT: $c \in \mathbb{Q}^n, x_0 \in \mathbb{Q}^n, \varepsilon, r, R \in \mathbb{Q}_+$: $B(x_0, r) \subseteq P \subseteq B(x_0, R)$
- OUTPUT: $y^* \in P$ with $c^T y^* \geq OPT_f - \varepsilon$

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   (5) *Case (A)* $\rightarrow$ *a:* Compute $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq a^T z_t\}$
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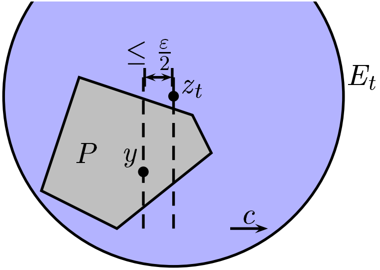
Case (B):



Grötschel-Lovász-Schrijver Algorithm

- INPUT: $c \in \mathbb{Q}^n, x_0 \in \mathbb{Q}^n, \varepsilon, r, R \in \mathbb{Q}_+$: $B(x_0, r) \subseteq P \subseteq B(x_0, R)$
- OUTPUT: $y^* \in P$ with $c^T y^* \geq OPT_f - \varepsilon$

(1) Ellipsod $E_0 := B(x_0, R)$ with center $z_0 := x_0$, $y^* := x_0$

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  (5) *Case (A)* $\rightarrow a$: Compute $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq a^T z_t\}$

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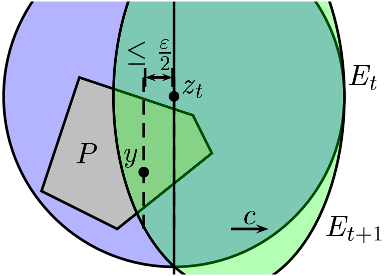
    (8) Compute $E_{t+1} \supseteq E_t \cap \{x \mid c^T x \geq c^T z_t\}$

Case (B):

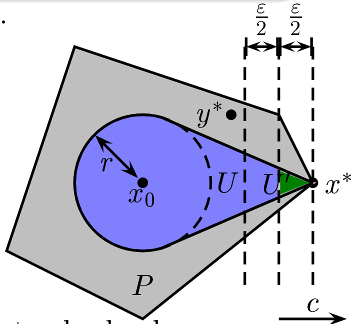


Analysis

Theorem

Let $OPT_f = \max\{c^T x \mid x \in P\}$. The GLS algorithm finds a $y^* \in P$ with $c^T y^* \geq OPT_f - \varepsilon$.

- Suppose for contradiction this is false.
- Let $x^* \in P$ be opt. sol.; φ input size.
- Inequalities from case (A) never cut points from P
- Ineq. from case (B) never cut points better than $OPT_f - \frac{\varepsilon}{2}$ (otherwise we would have found a suitable y^*)
- Let $U := \text{conv}\{B(x_0, r), x^*\}$ and $U' = \{x \in U \mid c^T x \geq OPT_f - \frac{\varepsilon}{2}\}$. By standard volume bounds: $\text{vol}(U') \geq (\frac{1}{2})^{poly(\varphi)}$. But $U' \subseteq E_t \; \forall t$. After $poly(\varphi)$ many it. $\text{vol}(E_t) = (1 - \frac{\Theta(1)}{n})^t \cdot \text{vol}(E_0) < \text{vol}(U')$. Contradiction!

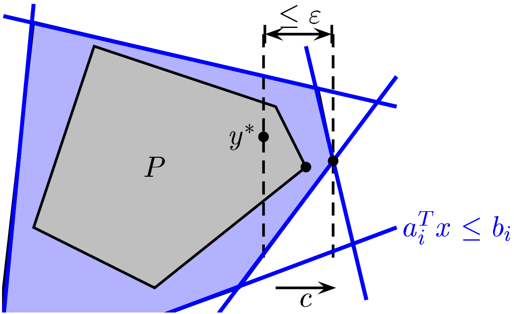


A useful observation

Observation

Consider a run of the GLS algorithm for $P \subseteq \mathbb{R}^n$ which yields $y^* \in P$. Let $a_1^T x \leq b_1, \ldots, a_N^T x \leq b_N$ be the inequalities which the oracle are returned for Case (A).

- Each $a_i^T x \leq b_i$ is feasible for P
- $c^T y^* \geq \max\{c^T x \mid a_i^T x \leq b_i \ \forall i = 1, \ldots, N\} - \varepsilon$



Solving $D(\mathcal{P})$

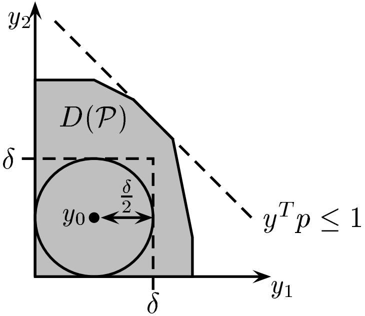
Lemma

Suppose $a_i \geq \delta$. Then we can find a feasible solution y^* to $D(\mathcal{P})$ of value $\geq OPT_f - 1$ in time polynomial in $n, m, \frac{1}{\delta}$.

- Apply GLS algo for $\varepsilon := 1$. Choose $y_0 = (\frac{\delta}{2}, \ldots, \frac{\delta}{2})$.

$$B\left(y_0, \frac{\delta}{2}\right) \overset{(\delta,\ldots,\delta)^T p \leq 1}{\subseteq} D(\mathcal{P}) \subseteq B(y_0, n)$$

- We use $\sum_{i=1}^{n} p_i \leq \frac{1}{\delta}$ for any feasible pattern $p \in \mathcal{P}$ since $a_i \geq \delta$

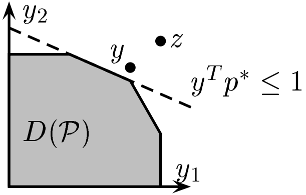


Solving $D(\mathcal{P})$ (2)

- We solve ε-weak separation problem for $z \in \mathbb{Q}^n$.
- If $z_i < 0 \rightarrow$ Case (A) (inequality $z_i \geq 0$ violated)
- If $z_i > 1 \rightarrow$ Case (A) (inequality $z^T e_i \leq 1$ violated)
- Round z down to nearest multiple of $\frac{1}{2m}$ and term this vector y. Solve $p^* = \text{argmax}\{y^T p \mid p \in \mathcal{P}\}$ (Knapsack with profits from $0, 1 \cdot \frac{1}{2m}, 2 \cdot \frac{1}{2m}, \ldots, 1$)

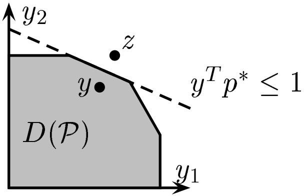
Case $y^T p^* > 1$:

- Then $z^T p^* \geq y^T p^* > 1$ $\rightarrow$ Case (A).



Case $y^T p^* \leq 1$:

- Then $y \in D(\mathcal{P})$. And $z^T b - y^T b \leq m \cdot \frac{1}{2m} = \frac{1}{2} = \frac{\varepsilon}{2}$. $\rightarrow$ Case (B)



- GLS yields a solution y^* mit $b^T y^* \geq OPT_f - 1$.

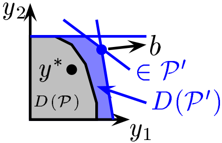
□

Finding a near optimal basic solution for $P(\mathcal{P})$

Theorem

Suppose $a_i \geq \delta$. Then we can find a basic solution x^* for $P(\mathcal{P})$ of value $\leq OPT_f + 1$ in time polynomial in $n, m, \frac{1}{\delta}$.

- Run GLS to obtain sol. y^* to $D(\mathcal{P})$ with $b^T y^* \geq OPT_f - 1$
- Let $y^T p \leq 1$, $p \in \mathcal{P}'$ be inequalities returned by oracle for case (A). $\mathcal{P}' \subseteq \mathcal{P}$ has polynomial size and



$$D(\mathcal{P}) \overset{y^* \text{ valid for } D(\mathcal{P})}{\geq} b^T y^* \geq D(\mathcal{P}') - 1 \qquad (1)$$

- Compute optimum basic solution x^* for $P(\mathcal{P}')$ in poly-time.

$$\mathbf{1}^T x^* = P(\mathcal{P}') \overset{\text{duality}}{=} D(\mathcal{P}') \overset{(1)}{\leq} D(\mathcal{P}) + 1 \overset{\text{duality}}{=} P(\mathcal{P}) + 1$$

- x^* is also a (non-optimal) basic solution for $P(\mathcal{P})$

Geometric Grouping

- INPUT: Instance $I = (a_1, \ldots, a_n)$, $size(I) = \sum_{i=1}^{n} a_i b_i \leq n$, $a_i \geq \delta$
- OUTPUT: Rounded up instance I' with $n/2$ diff. item sizes $OPT_f(I') \leq OPT_f(I)$ plus waste of $O(\log \frac{1}{\delta})$

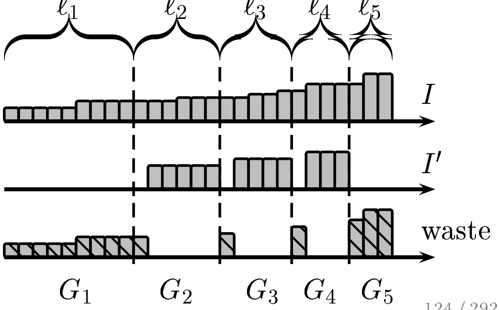
(1) Sort items w.r.t. sizes $e_1 \leq e_2 \leq \ldots \leq e_m$ (a_i appears b_i times)

(2) Let $G_1 = \{e_1, \ldots, e_{\ell_1}\}$ be minimal set of items with $\sum_{i \in G_1} e_i \geq 2$, then continue with $G_2, \ldots$. Let $\ell_i := |G_i|$ be number of items in G_i

(3) Remove first and last group → waste

(4) From G_i throw away smallest $\ell_i - \ell_{i+1}$ items → waste

(5) Round up items in G_i to largest item → I'



Geometric Grouping (2)

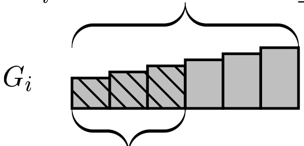
Lemma

Size of waste is $O(\log \frac{1}{\delta})$.

- Size of 1st and last group is $O(1)$
- Consider group G_i. Total size of items in G_i is ≤ 3.
- Num of groups is $\leq n/2$. Cleary $\frac{2}{\delta} \geq \ell_1 \geq \ell_2 \geq \ldots$.
- The $n_i := \ell_i - \ell_{i+1}$ smallest items in G_i have size $\leq 3\frac{n_i}{\ell_i}$.

$$\text{waste} \leq 3 \sum_i \frac{n_i}{\ell_i} \leq 3 \sum_{j=1}^{\ell_1} \frac{1}{j} \overset{\ell_1 \leq 2/\delta}{=} O(\log \frac{1}{\delta})$$

ℓ_i items of total size ≤ 3



n_i items of total size $\leq 3\frac{n_i}{\ell_i}$

The algorithm

Algorithm:

(1) Compute a basic solution x to $P(\mathcal{P})$ with $\mathbf{1}^T x \leq OPT_f + 1$
(2) Buy $\lfloor x_p \rfloor$ times pattern p, let I be remaining instance
(3) Apply geometric grouping to I (with n different item sizes) $\rightarrow I'$ (with $n/2$ different item sizes)
(4) Recurse

Theorem

One has $APX \leq OPT_f + O(\log^2 n)$.

- Since x is basic solution, $|\{p \mid x_p > 0\}| \leq n$.
- After (2) $size(I) \leq \sum_p (x_p - \lfloor x_p \rfloor) \leq n$.
- Let x^t be solution x in iteration t. We buy $\sum_p \lfloor x_p^t \rfloor$ bins, but OPT_f decreases by the same quantity.
- We pay in total $OPT_f +$ total waste. We have $O(\log n)$ recursions; in each recursion we have a waste of $O(\log \frac{1}{\delta}) = O(\log n)$. □

State of the art

- Computing OPT exactly is **NP**-hard even if the numbers a_i are unary encoded (i.e. BinPacking is strongly **NP**-hard).

Open question

One can compute a Bin Packing solution with $\leq OPT + 1$ bins in poly-time?

Mixed Integer Roundup Conjecture

One has $OPT \leq \lceil OPT_f \rceil + 1$.

Part 15
Minimum Makespan Scheduling

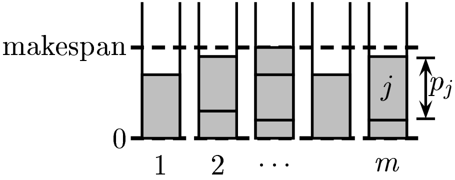
Source: *Approximation Algorithms* (Vazirani, Springer Press)

Minimum Makespan

Problem: Minimum Makespan Scheduling

- Given: n jobs, job j has processing time p_j. Number m of machines.
- Find: Assign jobs to machines to minimize the makespan.

$$OPT = \min_{I_1 \dot\cup \ldots \dot\cup I_m = \{1,\ldots,n\}} \left\{ \max_{i=1,\ldots,m} \left\{ \sum_{j \in I_i} p_j \right\} \right\}$$



A PTAS for Minimum Makespan Scheduling

Algorithm:

(1) Guess OPT

(2) Call job with $p_j > \varepsilon \cdot OPT$ large and small otherwise $\rightarrow$ sub-instance I of large jobs

(3) Round processing times p_j for large jobs down to multiple of $OPT \cdot \varepsilon^2 \rightarrow$ instance I' with processing times p'_j

(4) Distribute rounded large jobs I' such that makesepan is $\leq OPT$

(5) Distribute small jobs consecutively on least loaded machine

Analysis

Lemma

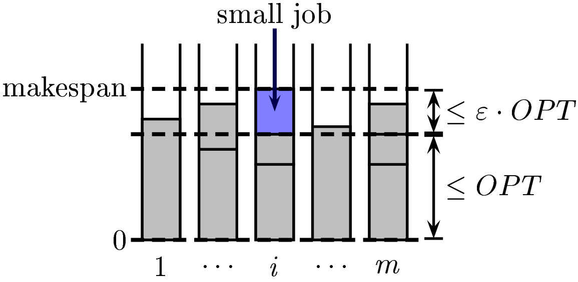
The algorithm runs in polynomial time and produces a makespan of at most $(1+\varepsilon)OPT$.

- Large jobs with rounded processing times can be distributed optimally in polynomial time since: $1/\varepsilon^2$ different job sizes, at most $1/\varepsilon$ large jobs per machine, hence $O((1/\varepsilon^2)^{1/\varepsilon})$ many ways how to pack a machine, hence $\leq n^{O((1/\varepsilon^2)^{1/\varepsilon})}$ possible solutions.
- Clearly $OPT(I') \leq OPT(I) \leq OPT$. Let I_i set of jobs on most loaded machine (attaining the makespan).
- *Case: Small jobs don't inc. makespan.* No small job in I_i.

$$\sum_{j \in I_i} p_j \leq \sum_{j \in I_i} (p'_j + \varepsilon \cdot \underbrace{\varepsilon OPT}_{\leq p'_j}) \overset{\sum_{j \in I_i} p'_j \leq OPT}{\leq} (1+\varepsilon)OPT$$

Analysis (2)

- $OPT \geq \frac{1}{m} \sum_{j=1}^{n} p_j =$ average load
- *Case: Small jobs do inc. makespan.* Then all machines are filled up to makespan $- \varepsilon \cdot OPT \leq OPT$. Hence makespan $\leq (1+\varepsilon)OPT$



Hardness

Lemma

There is no FPTAS for MINIMUM MAKESPAN SCHEDULING *unless* $\mathbf{NP} = \mathbf{P}$.

- Recall that given a BINPACKING instance $I = (a_1, \dots, a_n), a_i \in \mathbb{N}$ unary encoded and $m, B \in \mathbb{N}$, it is **NP**-hard to decide, whether m bins of size B suffice to pack the items.
- Suppose there is an FPTAS for MINIMUM MAKESPAN SCHEDULING. Take items as jobs, m as number of machines and $\varepsilon := \frac{1}{\sum_{i=1}^{n} a_i + 1}$. Then the FPTAS would give an exact answer.

opt. makespan $\leq B \Leftrightarrow \exists$ BIN PACKING solution with m bins.

Part 16
Scheduling on Unrelated Parallel Machines

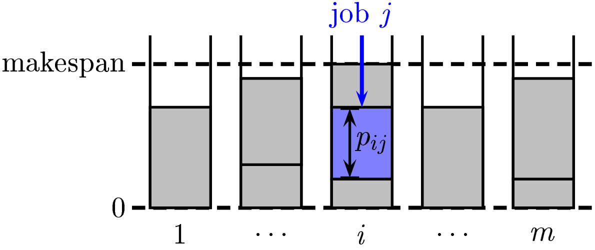
Source: *Approximation Algorithms* (Vazirani, Springer Press)

Scheduling on Unrelated Parallel Machines

Problem: UNRELATED MACHINE SCHEDULING

- Given: Jobs $J = \{1, \ldots, n\}$, machines $M = \{1, \ldots, m\}$. Running job j on machine i takes a processing time p_{ij}.
- Find: Assign jobs to machine to minimize the makespan.

$$OPT = \min_{I_1 \dot\cup \ldots \dot\cup I_m = \{1,\ldots,n\}} \left\{ \max_{i=1,\ldots,m} \left\{ \sum_{j \in I_i} p_{ij} \right\} \right\}$$



How NOT to solve the problem

LP:

$$\min T$$

$$\begin{aligned} \sum_{i \in M} x_{ij} &= 1 \quad \forall j \in J \\ \sum_{j \in J} p_{ij} x_{ij} &\leq T \quad \forall i \in M \\ x_{ij} &\geq 0 \quad \forall i\, \forall j \end{aligned}$$

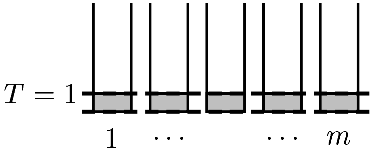
Variables:

$$x_{ij} = \begin{cases} 1 & \text{job } j \text{ is assigned} \\ & \text{to machine } i \\ 0 & \text{otherwise} \end{cases}$$

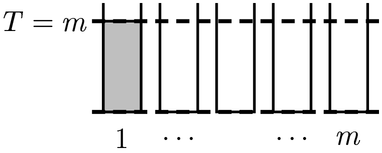
$$T = \text{makespan}$$

Example: 1 job with execution time $p_{i1} = m$, $\forall i = 1, \ldots, m$

Fractional solution: $x_{i1} = \frac{1}{m}$



Integer solution: $x_{11} = 1$



- Integrality gap of $\geq m$

A 2-approximation

Algorithm:

(1) Guess OPT

(2) Compute basic solution x^* to

$$
\begin{aligned}
\sum_{i \in M} x_{ij} &= 1 && \forall j \in J \\
\sum_{j \in J} p_{ij} x_{ij} &\leq OPT && \forall i \in M \\
x_{ij} &= 0 && \text{for } i, j \text{ with } p_{ij} > OPT \\
x_{ij} &\geq 0 && \forall i \in M \; \forall j \in J
\end{aligned}
$$

(3) $x^*_{ij} = 1 \Rightarrow$ assign job j to machine i

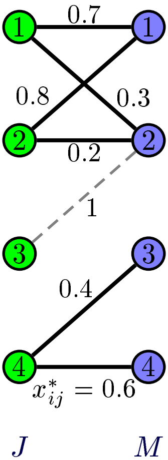
(4) For not yet assigned jobs: Assign j to a machine i with $0 < x^*_{ij} < 1$ s.t. every machine receives at most 1 extra job

Analysis

Theorem

The algorithm runs in polynomial time and the makespan is at most $OPT + \max\{p_{ij} \mid x^*_{ij} > 0\} \leq 2 \cdot OPT$.

- Running time is clearly polynomial: We solve a poly size LP in (2) and solve a maximum matching problem in (4).
- Let $H = (J \cup M, E)$ with $E := \{(j, i) \mid 0 < x^*_{ij} < 1\}$. For claim on makespan we need to show that E contains a $\{j \text{ not assigned in (3)}\}$-perfect matching.



Analysis

Theorem

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- Running time is clearly polynomial: We solve a poly size LP in (2) and solve a maximum matching problem in (4).
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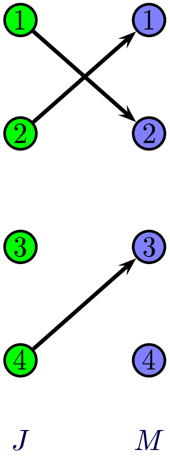
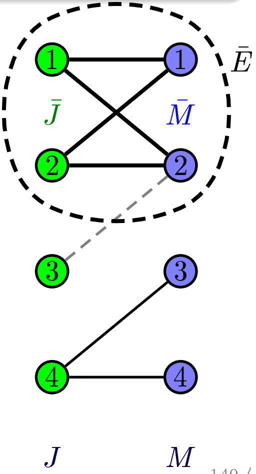


Assigning the fractional jobs (1)

Claim

Consider a connected component $(\bar{J} \cup \bar{M}, \bar{E})$ of H. Then $\bar{x}^* = (x^*_{ij})_{(j,i)\in\bar{E}}$ is still a basic solution of the subsystem $LP(\bar{E})$.

$$\begin{aligned}
\sum_{i\in\bar{M}} x_{ij} &= 1 \quad \forall j \in \bar{J} \quad (LP(\bar{E})) \\
\sum_{j\in\bar{J}} p_{ij}x_{ij} &\leq T - \sum_{j\notin\bar{J}} p_{ij}x^*_{ij} \quad \forall i \in \bar{M} \\
0 \leq x_{ij} &\leq 1 \quad \forall (j,i) \in \bar{E}
\end{aligned}$$



Reason: If $\bar{x}^* \in \text{conv}(\{y^{(1)}, y^{(2)}\})$ then $x^* = (\bar{x}^*, \hat{x}^*) \in \text{conv}(\{(y^{(1)}, \hat{x}^*), (y^{(2)}, \hat{x}^*)\})$.
Contradiction.

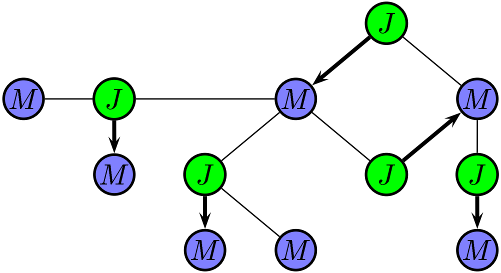
Assigning the fractional jobs (2)

- $\bar{x}$ is basic solution, hence

$$|\bar{E}| = |\{(j,i) \mid 0 < \bar{x}^*_{ij} < 1\}| \leq \overbrace{|\bar{J}| + |\bar{M}|}^{=\#\text{constr. in } LP(\bar{E})} \leq \#\text{nodes in } \bar{E}$$

- But $\bar{E}$ is connected, thus $\bar{E}$ is a tree + ≤ 1 extra edge.
- Jobs have degree ≥ 2, hence leaves must be machines. As long as there are machine-leaves i, assign a j with $x_{ij} > 0$ to i and remove both, i and j.
- A single even length job-machine cycle (potentially) remains. Extract a matching and we are done.

$\bar{E}$:



State of the art

Exercise

There is no $(3/2 - \varepsilon)$-apx for Unrelated Machine Scheduling unless $\mathbf{NP} = \mathbf{P}$.

Open Problem 1

Is there a 3/2-apx?

Open Problem 2

A $(2 - \varepsilon)$-apx is still unknown even for the Restricted Assignment Problem where $p_{ij} \in \{p_j, \infty\}$.

Theorem (Ebenlendr, Krcal, Sgall '08)

There is a 1.75-apx for the Restricted Assignment Problem if each job j is admissible on ≤ 2 machines.

Part 17
Multiprocessor Scheduling with Precedence Constraints

Source:

- Graham (1966): Bounds for certain multiprocessor anomalies (Bell Systems Technical Journal).
- Lecture notes of Chandra Chekuri http://www.cs.illinois.edu/class/sp09/cs598csc/Lectures/lecture˙6.pdf

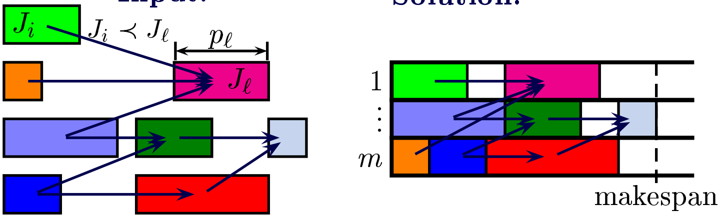
Multiprocessor Scheduling with Precedence Constraints

Problem: PrecScheduling ($P \mid p_i, \text{prec} \mid C_{\max}$)

- Given: Jobs $J_1, \ldots, J_n$, job J_i has processing time p_i, precedence relation $\prec$, # of machines m
- Find: (Non-preemptive) schedule of the jobs on m machines respecting the precedence order and minimizing the makespan

- $J_i \prec J_\ell$ means that J_i has to be finished, before J_ℓ is allowed to start.

Input: **Solution:**



The algorithm

Graham's List Scheduling:

(1) FOR $t = 1, \ldots$ DO

  (2) IF a machine $j \in \{1, \ldots, m\}$ is idle at t
  AND all predecessors of some (not yet processed) job J_i are already finished
  THEN schedule J_i on machine j starting from t

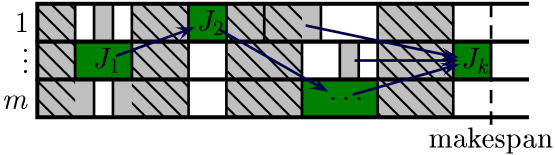
- In other words: At any time, just start a job whenever possible.

The analysis

Theorem

The makespan of the produced schedule is at most $2 \cdot OPT$

- Find a sequence (w.l.o.g. after reordering) $J_1, \ldots, J_k$ s.t.
  - J_k is the last job of the whole schedule that finishes
  - $J_1 \prec J_2 \prec \ldots \prec J_k$ (chain in the partial order $\prec$)
  - J_i is the predecessor of J_{i+1} that is finished last



- After J_i finished J_{i+1} is startet as soon as a machine is available. Hence between J_i is finished and J_{i+1} begins, **all** machines must be fully busy.
- length of all busy periods $\leq OPT$
- Length of chain $J_1, \ldots, J_k$ is $\leq OPT$
- Makespan $\leq$ length chain + busy period $\leq 2 \cdot OPT$

Hardness

Theorem (Svensson - STOC'10)

For every fixed $\varepsilon > 0$, there is no $(2-\varepsilon)$-apx unless a variant of the **Unique Games Conjecture** *is false.*

Open Problem

What is the complexity status of $P3 \mid p_i = 1, \text{prec} \mid C_{\max}$ (i.e. PRECSCHEDULING with unit processing times and 3 machines)?
Known:

- $4/3$-apx.
- $P2 \mid p_i = 1, \text{prec} \mid C_{\max}$ is poly-time solvable

Part 18
Euclidean TSP

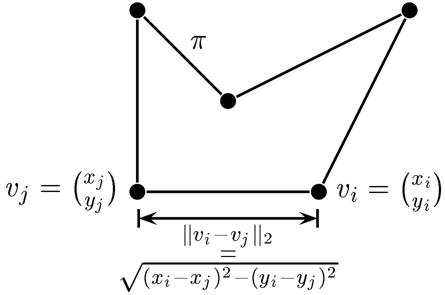
Source: *Polynomial-time Approximation Schemes for Euclidean TSP and other Geometric Problems* (Arora '98, Link)

Euclidean Travelling Salesman Problem

Problem: EUCLIDEANTSP

- Given: Points $v_1, \dots, v_n \in \mathbb{Q}^2$ in the plane.
- Find: Minimum cost tour visiting all nodes

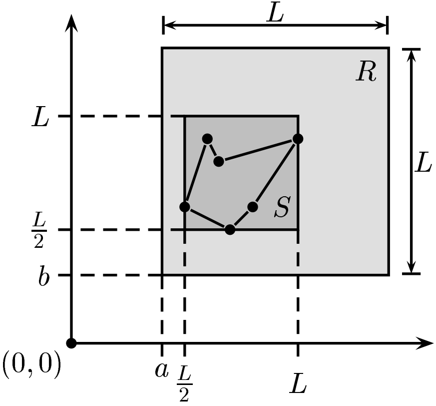
$$\min_{\text{tour } \pi: V \to V} \left\{ \sum_{i=1}^{n} \|v_i - v_{\pi(i)}\|_2 \right\}$$



Goal: Find a PTAS!

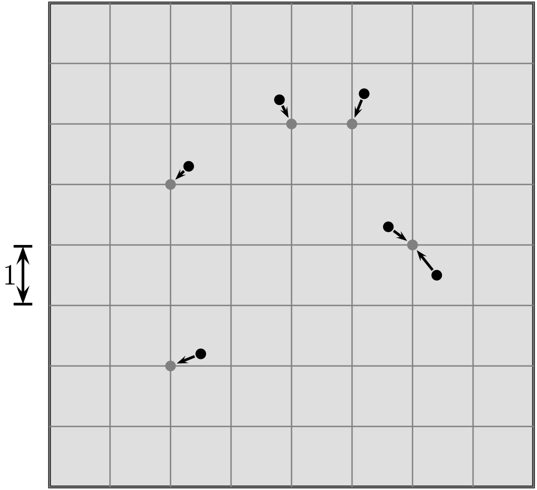
A random bounding box

- Choose a minimal square S containing all points.
- W.l.o.g. this square is $[\frac{L}{2}, L]^2$ with $L = n/\varepsilon \in 2^{\mathbb{N}}$ after scaling. Hence $OPT \geq L = n/\varepsilon$.
- Choose $a, b \in \{1, \ldots, L/2\}$ randomly.
- Let $R = [a, a+L] \times [b, b+L] \supseteq S$ be the randomly shifted bounding box.



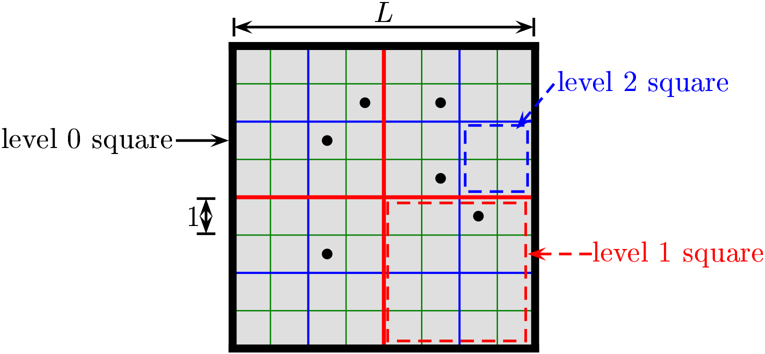
Discretization

- Move all points v to nearest point in $\mathbb{Z}^2$.
- Changes the cost of any tour by $\leq 2n \leq 2\varepsilon \cdot OPT$ (since $OPT \geq L = n/\varepsilon$)



The dissection

- Divide the $L \times L$ bounding box into 4 squares of size $\frac{L}{2} \times \frac{L}{2}$
- Divide each $\frac{L}{2} \times \frac{L}{2}$ square into 4 squares of size $\frac{L}{4} \times \frac{L}{4}$
- Recurse, until unit size squares are reached
- Size $\frac{L}{2^i} \times \frac{L}{2^i}$ squares are level i squares
- A line segment is on level i, if it is the boundary of a level i square but not of a level $i-1$ square
- A grid line is on level i, if it consists of level i segments



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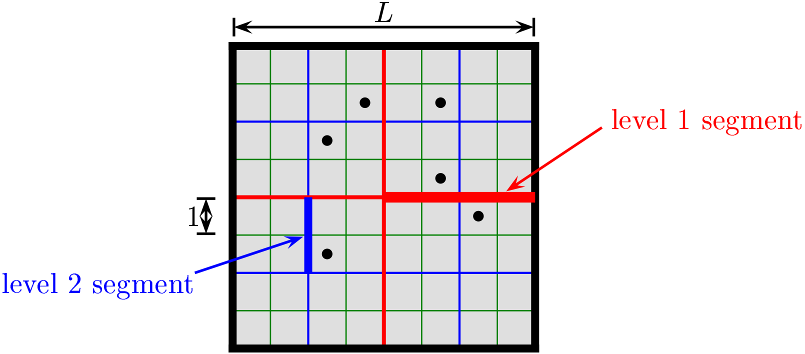

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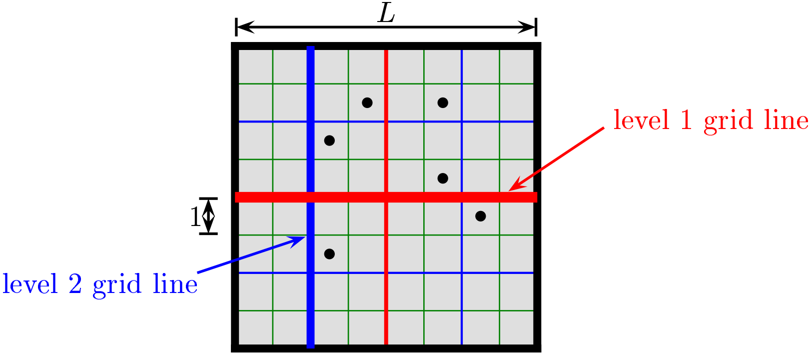
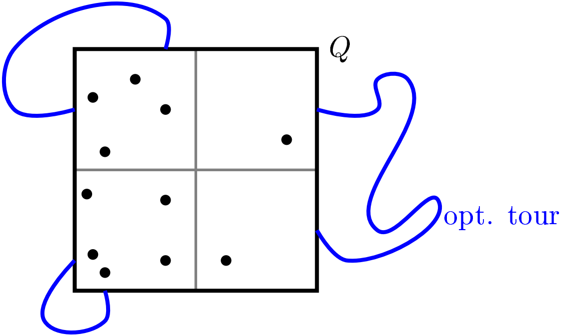


Basic idea

- **Method:** Use dynamic programming.
- **Idea:** Consider a level i square Q in the dissection. For all ways how OPT can intersect Q, compute the cheapest extension inside Q that visits all nodes in Q (using that we computed similar information already for all smaller squares).



- **Difficulty:** The number of possibilities how OPT can cross Q might be exponential/infinite.
- **Solution:** Limit this number.

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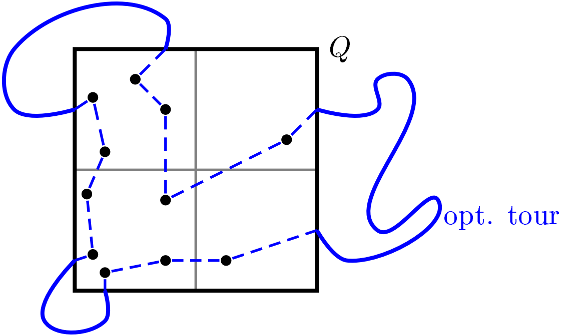
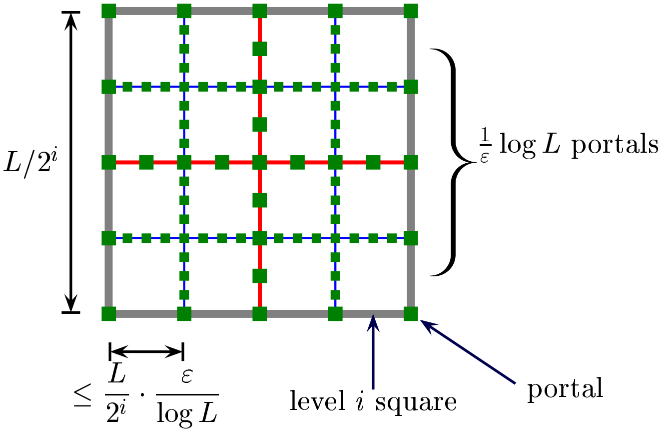


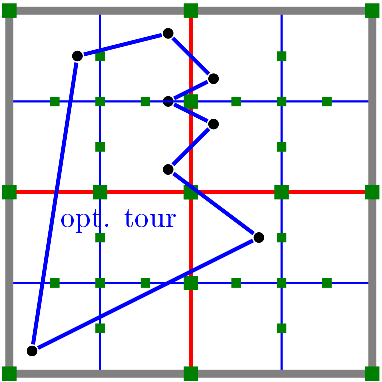
- **Difficulty:** The number of possibilities how OPT can cross Q might be exponential/infinite.
- **Solution:** Limit this number.

Portals

- On any level i line segment, place $\frac{1}{\varepsilon} \log L$ many level i portals (plus one per corner)
- Distance of consecutive level i portals is $\leq \frac{L}{2^i} \cdot \frac{\log L}{\varepsilon}$



Well rounded tours



Definition

A tour π is called well-rounded tour if:

- It leaves and enters squares only at portals.
- Each square is entered at most $\frac{4}{\varepsilon}$ times.

- Each square has $\leq \frac{4}{\varepsilon} \log L + 4$ many portals. The number of times that a well-rounded tour can leave/enter a square is bounded by $\leq (\frac{4}{\varepsilon} \log L + 4)^{O(1/\varepsilon)}$ (which is polynomial).

Theorem (Structure Theorem)

There is always a well-rounded tour of cost $\leq (1 + O(\varepsilon))OPT$.

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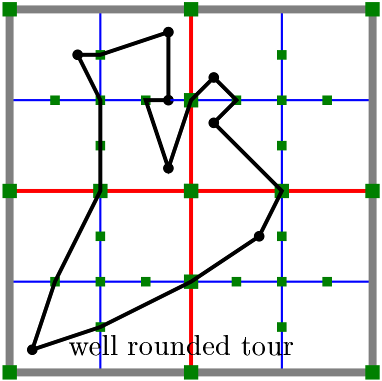

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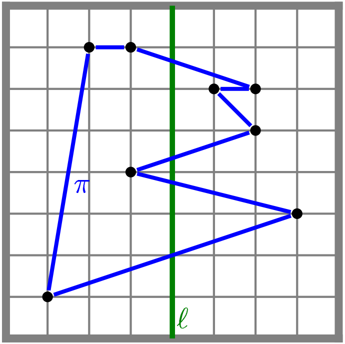
Relation OPT vs. number of crossings

- For the optimum tour π and a grid line ℓ, let $t(\pi, \ell)$ be the number of times that π crosses ℓ.

$$\frac{1}{3} \cdot \sum_{\text{grid lines } \ell} t(\pi, \ell) \leq OPT \leq \sqrt{2} \cdot \sum_{\text{grid lines } \ell} t(\pi, \ell)$$

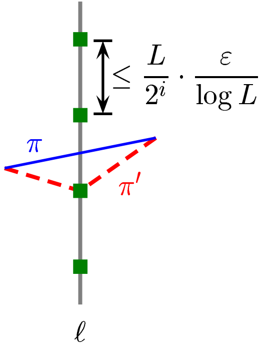
- $OPT = \Theta(1) \cdot \#\text{crossings}$
- **Goal:** Turn opt. tour π into a well-rounded tour, such that the expected cost increase is $O(\varepsilon) \cdot \sum_\ell t(\pi, \ell)$
- **Alternatively:** Average cost increase per crossing must be $O(1) \cdot \varepsilon$



$t(\pi, \ell) = 4$

Bending edges through portals

- Consider a crossing of the optimum tour π at a grid line ℓ
- $\Pr[\text{line } \ell \text{ is at level } i] = \dfrac{2^i}{L}$
- If line ℓ is at level i, we have to bend edge through the nearest portal and loose $\leq \dfrac{L}{2^i} \cdot \dfrac{\varepsilon}{\log L}$
- The expected length increase is



$$\sum_{i=0}^{\log L} \Pr[\ell \text{ at level } i] \cdot \text{portal distance at level } i$$

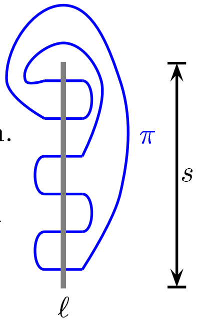
$$= \sum_{i=0}^{\log L} \frac{2^i}{L} \cdot \frac{L}{2^i} \cdot \frac{\varepsilon}{\log L} \leq 2\varepsilon$$

Patching Lemma

Lemma

Given a TSP tour π, crossing a line segment ℓ of length s an arbitrary number of times. $\exists$ tour π' crossing ℓ at most 2 times which can be obtained by adding segments of length $\leq 6s$.

- Cut π at ℓ. Let $L_1, \ldots, L_t$ be endpoints on the left side, $R_1, \ldots, R_t$ end points on the right. Imagine their distance to ℓ as 0. Say t is even (other case is similar).
- Add tours on L_i's and on R_i's of cost $\leq 2s$ each.
- Add matchings $(L_{2i-1}, L_{2i}), (R_{2i-1}, R_{2i})$ for $2i < t$ and 2 edges $(L_{t-1}, R_{t-1}), (L_t, R_t)$ of total cost $\leq 2s$.
- Degree of $V \cup \{L_i, R_i \mid i = 1, \ldots, t\}$ is even. Graph is again connected. Hence there is a tour visiting all nodes (at least once).



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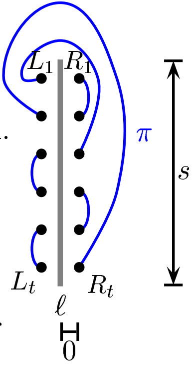

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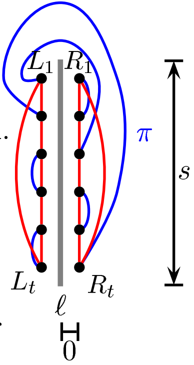

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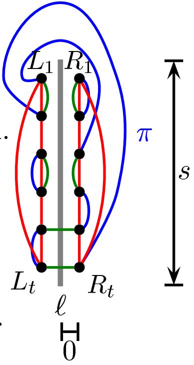


Reducing the number of crossings (1)

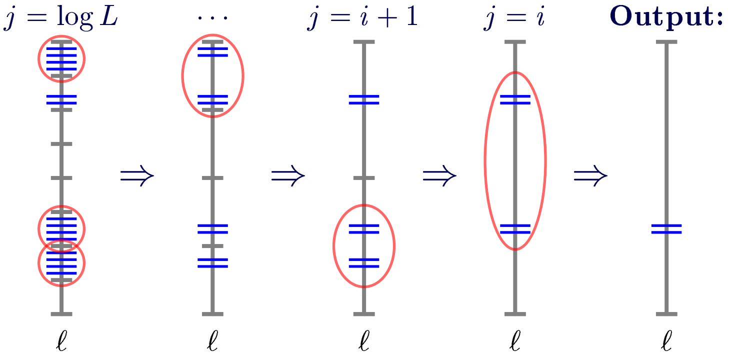
`MODIFY` **Procedure:**

- **Input:** Grid line ℓ on level i
- **Output:** Tour π' crossing each segment of ℓ at most $1/\varepsilon$ times

(1) FOR $j = \log L$ downto i DO

  (2) FOR all level j segments DO

    (3) IF segment is crossed $> 1/\varepsilon$ times
    THEN reduce # crossings to 2 via Patching Lemma



Reducing the number of crossings (2)

- Starting from optimum tour, we apply `MODIFY` to all horizontal and vertical grid lines.
- Now consider a fixed grid line ℓ. Want to show:

$$E[\text{cost for crossing reduction at } \ell] \leq O(\varepsilon) \cdot t(\pi, \ell)$$

- Let $c_{\ell,j}$ be number of times that `MODIFY` is applied to level j segments of grid line ℓ
- Each application of `MODIFY` reduces the number of crossings of ℓ by $1/\varepsilon - 2 \geq \frac{1}{2\varepsilon}$ (assuming $\varepsilon \leq 1/4$). Hence

$$\sum_{j \geq 0} c_{\ell,j} \leq \frac{t(\pi, \ell)}{1/(2\varepsilon)} = 2\varepsilon \cdot t(\pi, \ell)$$

- The cost increase of a single crossing reduction on level j is $\leq 6 \cdot \frac{L}{2^j}$ (by Patching Lemma).
- Thus

$$E[\text{cost increase at } \ell \mid \ell \text{ at level } i] \cdot \leq \sum_{j \geq i} c_{\ell,j} \cdot 6 \cdot \frac{L}{2^j}$$

Reducing the number of crossings (3)

$$
\begin{aligned}
& E[\text{cost for crossing reduction at } \ell] \\
= \quad & \sum_{i\geq 0} \Pr[\ell \text{ at level } i] \cdot E[\text{cost increase at } \ell \mid \ell \text{ at level } i] \\
\leq \quad & \sum_{i\geq 0} \frac{2^i}{L} \cdot \sum_{j\geq i} c_{\ell,j} \cdot 6\frac{L}{2^j} \\
\overset{\text{reordering}}{=} \quad & 6 \sum_{j\geq 0} \frac{c_{\ell,j}}{2^j} \cdot \underbrace{\sum_{i\leq j} 2^i}_{\leq 2\cdot 2^j} \\
\leq \quad & 12 \cdot \sum_{j\geq 0} c_{\ell,j} \\
\overset{\sum_{j\geq 0} c_{\ell,j} \leq 2\varepsilon\cdot t(\pi,\ell)}{\leq} \quad & 24\varepsilon \cdot t(\pi, l)
\end{aligned}
$$

- $\exists$ well-rounded tour of cost $(1 + O(\varepsilon)) \cdot OPT$ □

The dynamic program (1)

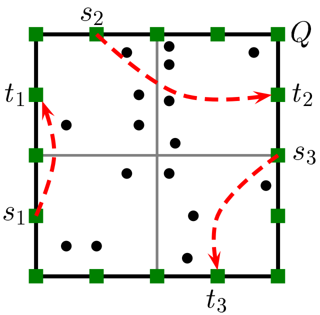
- **Table entries:**

$$A(Q,(s_1,t_1),\dots,(s_q,t_q))$$
$$= \text{cost of cheapest extension of } q \text{ subtours to well-rounded tour visiting all nodes in } Q \text{ such that subtour } i \text{ goes from } s_i \text{ to } t_i$$

$\forall$ squares Q $\forall q \in \{0,\dots,4/\varepsilon\}$ $\forall$ portals s_i, t_i of Q

- **Number of table entries:**
  - $O(n \cdot \log L)$ many non-empty squares Q
  - There are $O(\frac{1}{\varepsilon}\log n)^{O(1/\varepsilon)}$ many ways to choose $O(1/\varepsilon)$ portals out of $O(\frac{1}{\varepsilon}\log L)$ portals
  - Total number of entries:

$$O(n(\log n)^{O(1/\varepsilon)})$$



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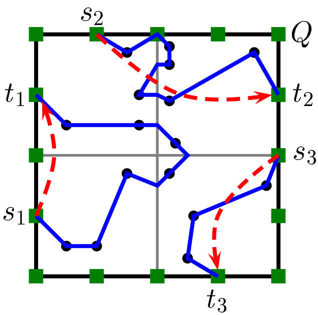
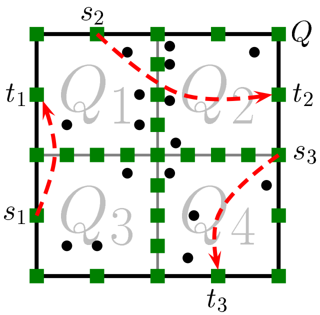


The dynamic program (2)

Lemma

The best well rounded tour can be computed in $O(n(\log n)^{O(1/\varepsilon)})$

- Compute table entries bottom-up (starting with smallest squares)
- For entry $A(Q,(s_1,t_1),\ldots,(s_q,t_q))$: Let $Q_1,\ldots,Q_4$ be the subsquares of Q. Guess (i.e. try out all combinations) the visited portals of $Q_1,\ldots,Q_4$ and their order $\rightarrow O(\frac{1}{\varepsilon}\log n)^{O(1/\varepsilon)}$ combinations



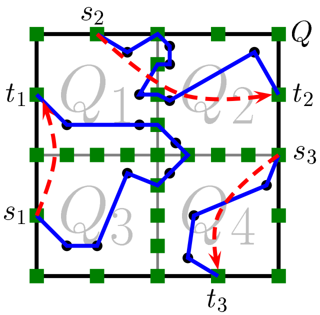
- Look up table entries for $Q_1,\ldots,Q_4$ to determine cost.

The dynamic program (2)

Lemma

The best well rounded tour can be computed in $O(n(\log n)^{O(1/\varepsilon)})$

- Compute table entries bottom-up (starting with smallest squares)
- For entry $A(Q, (s_1, t_1), \ldots, (s_q, t_q))$: Let $Q_1, \ldots, Q_4$ be the subsquares of Q. Guess (i.e. try out all combinations) the visited portals of $Q_1, \ldots, Q_4$ and their order $\rightarrow O(\frac{1}{\varepsilon} \log n)^{O(1/\varepsilon)}$ combinations
- Look up table entries for $Q_1, \ldots, Q_4$ to determine cost.



Generalizations

Advantages of this approach:

- Applicable for many graph optimization problems, when nodes are points in the Euclidean plane (like Steiner Tree, k-Median, Steiner Forest, k-Tsp, k-Mst.
- Works for general ℓ_p-metrices (like maximums-norm)
- Extends to any constant dimension
- (Theoretically) nice dependence on ε

Theorem (Arora '98)

Let $d \in \mathbb{N}, \varepsilon > 0, p \in \mathbb{N} \cup \{\infty\}$ be fixed constants. Then there is an expected $(1+\varepsilon)$-apx for TSP *if the nodes are points in $\mathbb{R}^d$ and distances are measured as $\|v-u\|_p := (\sum_{i=1}^d |v_i - u_i|^p)^{1/p}$ in time $n(O(\log n))^{O(\sqrt{d} \cdot 1/\varepsilon)^{d-1}}$. This can be derandomized by increasing the running time by a factor of $O(n/\varepsilon)$.*

Part 19
Tree Embeddings

Source: *A tight bound on approximating arbitrary metrics by tree metrics* (Fakcharoenphol, Rao, Talwar: Link)

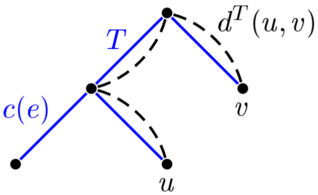
Tree metric

Definition (Tree metric)

Given nodes V, spanning tree T, edge costs $c(e)\ \forall e \in T$. Then $d^T : V \times V \to \mathbb{Q}_+$ with

$$d^T(u,v) := \text{length of } u - v \text{ path in } T$$

is called a tree metric.

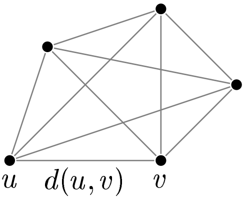


Motivation

- **Motivation:** Many optimization problems are easy on trees: Steiner tree, Tsp, k-Tsp, Steiner Forest, ...
- **Question:** Can we for any node set V and metric $d : V \times V \to \mathbb{Q}_+$, find a tree metric d^T such that

$$d(u,v) \leq d^T(u,v) \leq \alpha \cdot d(u,v) \quad \forall u,v \in V$$

  for a small distortion α?



- **Possible approach:** For some graph optimization problem, compute tree T. Then solve problem on tree optimally (or get $O(1)$-apx). Obtain a α-apx (or $O(\alpha)$-apx) for original problem.

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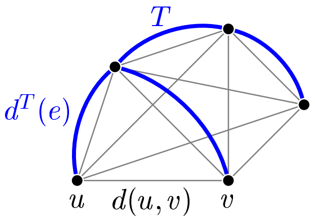

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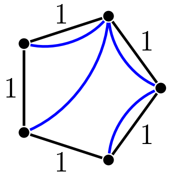
One good, one bad news

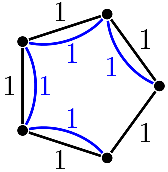
Bad news:

Theorem (Rabinovitch, Raz '95)

Any tree embedding for an n-cycle must have distortion $\Omega(n)$.



Good news:



- Delete a random edge.
- For $u, v \in V$ with $d(u,v) = k$ one has $d^T(u,v) = n - k$ with probability $\frac{k}{n}$ and $d^T(u,v) = k$ with probability $1 - \frac{k}{n}$.
- Expected distortion is at most 2 since:

$$E[d^T(u,v)] = \underbrace{\frac{k}{n}(n-k)}_{\leq k} + \underbrace{\left(1 - \frac{k}{n}\right) \cdot k}_{\leq k} \leq 2 \cdot k$$

The Theorem

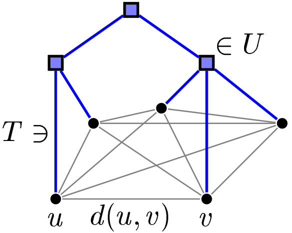
Theorem (Fakcharoenphol, Rao, Talwar '03)

Given any metric (V, d), one can find randomly (in time $O(n^2)$) a tree metric $(V \cup U, d^T)$ such that

- $d(u, v) \leq d^T(u, v)\ \forall u, v \in V$ *(i.e. d^T dominates d)*
- $E[d^T(u, v)] \leq O(\log n) \cdot d(u, v)\ \forall u, v \in V$

That means the tree metric has an expected $O(\log n)$ distortion.

Remark: The tree will contain extra nodes U, which were not contained in the original nodeset.



Preliminaries

Assumptions:

- $2^\delta = \max_{u,v \in V}\{d(u,v)\}$ is diameter
- $d(u,v) > 1 \; \forall u \neq v$

Definition

A set system $\mathcal{S}$ is called laminar if for every $S_1, S_2 \in \mathcal{S}$ one has either $S_1 \cap S_2 = \emptyset$ or $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

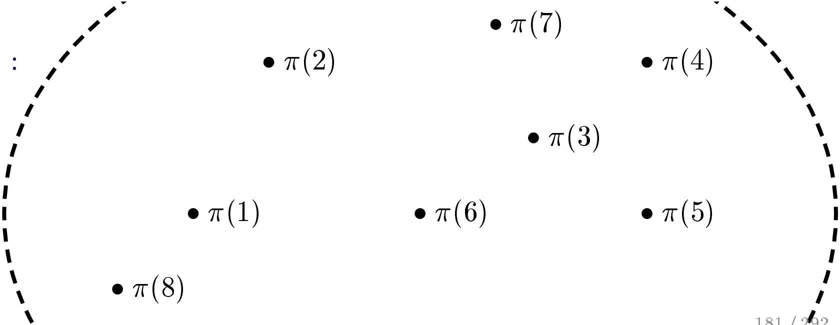
Idea: Obtain a random laminar family.

Clustering

Algorithm:

(1) Choose a random permutation π on nodes V
(2) Choose $\beta \in [0,1]$ uniformly at random
(3) $D_\delta := \{V\}$
(4) FOR $i = \delta - 1$ DOWNTO 0 DO
    (5) Assign every node to first node (w.r.t. order π) that has distance $\leq 2^\beta \cdot 2^{i-1}$
    (6) All nodes that are assigned to the same node and are in the same cluster (in D_{i+1}) form a new cluster of D_i

D_δ :



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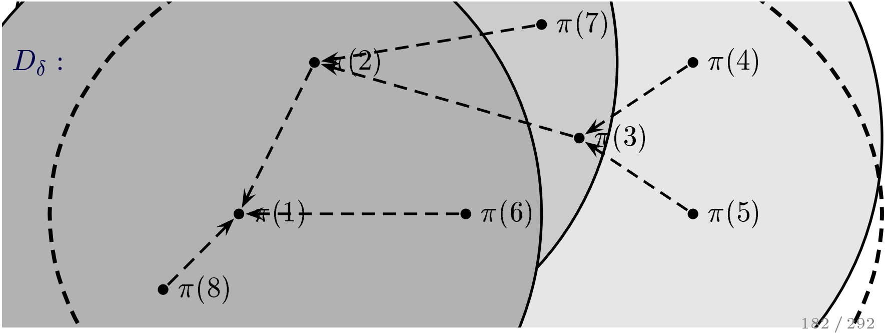

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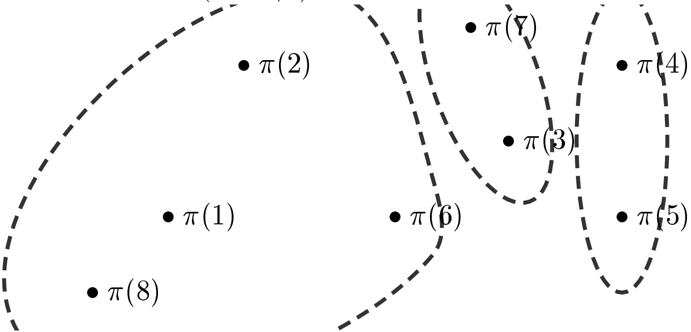

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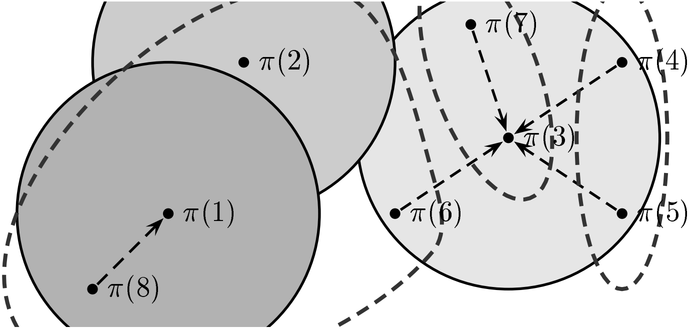

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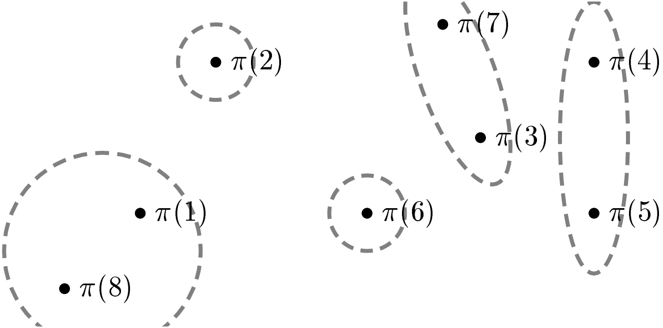

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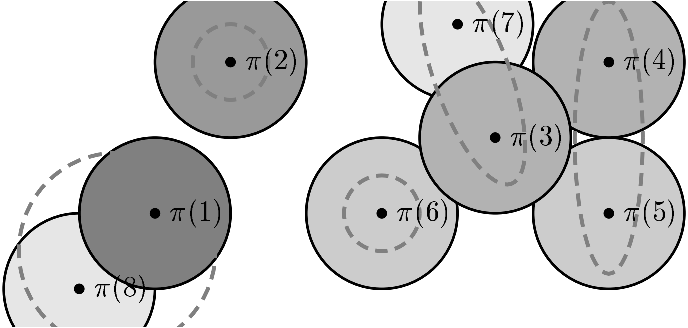

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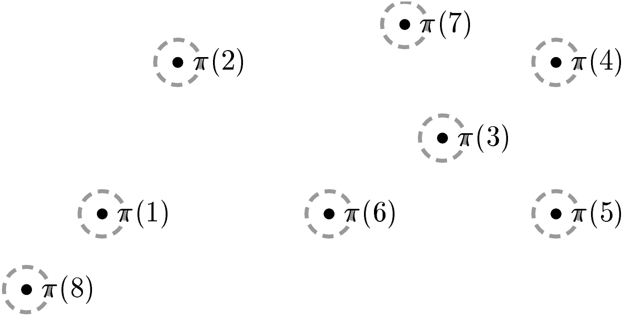
D_0 :



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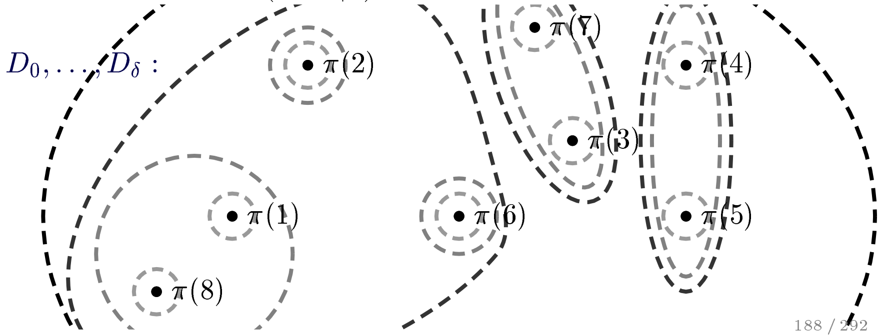
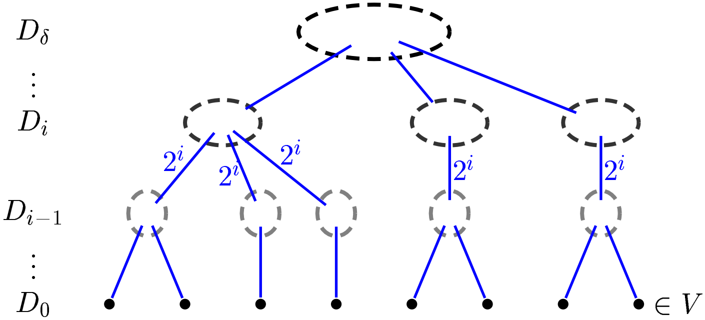


Defining the tree metric

- Each cluster becomes an extra node
- Insert edge of cost 2^i between $S \in D_i, S' \in D_{i-1}$ if $S' \subseteq S$



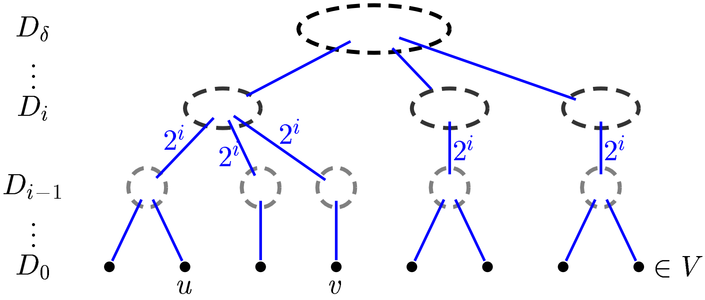
- Note that in the last iteration ($i = 0$) we assign each node to a cluster center at distance $\leq 2^\beta \cdot 2^{0-1} \leq 1$. Hence the clusters of D_0 are indeed singletons (since $d(u, v) > 1 \; \forall u \neq v$).

d^T dominates d

Lemma

The tree metric d^T dominates d, i.e. $d(u,v) \leq d^T(u,v) \forall u,v \in V$

- Suppose u, v are in the same D_i cluster, but separated by D_{i-1}
- Cluster in D_i have diameter $\leq 2 \cdot 2^\beta \cdot 2^{i-1} \leq 2^{i+1}$
- On the other hand $d^T(u,v) \geq 2 \cdot 2^i$.
- Hence $d(u,v) \leq 2^{i+1} \leq d^T(u,v)$ □



Proof of $O(\log n)$ average distortion

Lemma

For any $u, v \in V$: $E[d^T(u,v)] = O(\log n) \cdot d(u,v)$

- If only one of the nodes u, v is assigned to center w in an iteration i, then we say w cuts edge (u,v) at level i.
- We want to charge the u-v distance to that cluster center that cuts the u-v edge

$$d_w^T(u,v) := \sum_{i:w \text{ cuts } (u,v) \text{ at level } i} 2^{i+2}$$

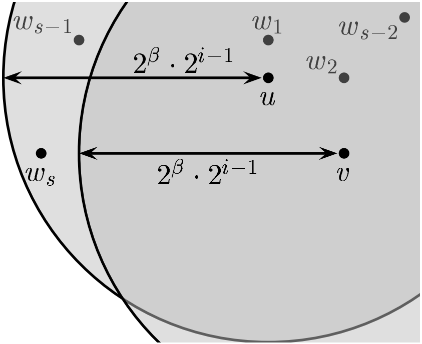
- Then

$$d^T(u,v) \leq \sum_{w \in V} d_w^T(u,v)$$

since: Suppose u, v are separated by D_i (i.e. they are in the same D_{i+1} cluster). Then $d^T(u,v) \leq \sum_{j=0}^{i+1} 2 \cdot 2^j \leq 2 \cdot 2^{i+2}$. But in iteration i, we find 2 cluster centers w, w' that cut edge (u,v), for both $d_w^T(u,v), d_{w'}^T(u,v) \geq 2^{i+2}$.

Proof of $O(\log n)$ average distortion (2)

- Assume w.l.o.g. that $d(u, w_s) < d(v, w_s)$.
- Let $w_1, w_2, \ldots$ be nodes in increasing distance from u
- w_s can cut (u, v) only if
  - **(A)** $\exists$ level i, where $d(u, w_s) \leq 2^\beta \cdot 2^{i-1} < d(v, w_s)$
  - **(B)** u is assigned to w_s



Proof of $O(\log n)$ average distortion (3)

- Assume for a second: $\exists i : 2^{i-1} \leq d(u, w_s) < d(v, w_s) < 2^i$.
- Then there is only one level i at which w_s might cut (u, v)
- By triangle inequality, the length of the interval $[d(u, w_s), d(v, w_s)]$ is

$$d(v, w_s) - d(u, w_s) \leq d(u, v).$$

- Logscale length of interval is at most $\log_2\left(\frac{2^{i-1}+d(u,v)}{2^{i-1}}\right)$.

$$\Pr[(A)] \leq \log_2\left(\frac{2^{i-1} + d(u,v)}{2^{i-1}}\right) \overset{\log_2(1+x)\leq 2x}{\leq} 2 \cdot \frac{d(u,v)}{2^{i-1}}$$

Standard:

Logscale:

Proof of $O(\log n)$ average distortion (3)

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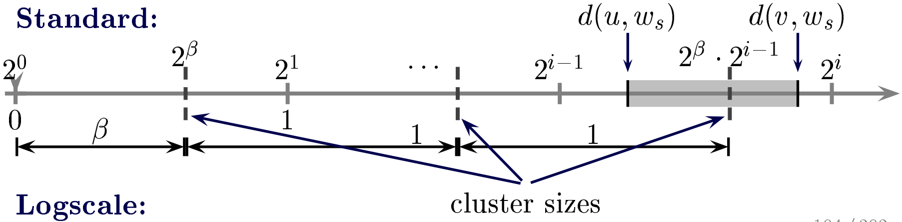


Proof of $O(\log n)$ average distortion (4)

- Next, condition on (A).

$$\Pr[u \text{ assigned to } w_s | (A)] \leq \Pr[w_s \text{ 1st of } w_1, \ldots, w_s \text{ w.r.t. } \pi] = \frac{1}{s}$$

- If (A) & (B) happen, this incurs cost of 2^{i+2}.
- Hence

$$E[d^T_{w_s}(u,v)] \leq 2^{i+2} \cdot 2 \cdot \frac{d(u,v)}{2^{i-1}} \cdot \frac{1}{s} = O\Big(\frac{d(u,v)}{s}\Big)$$

- For general case: Let δ_i be length of $[d(u,w_s), d(v,w_s)] \cap [2^{i-1}, 2^i]$ Then applying the arguments for each δ_i: $E[d^T_{w_s}(u,v)] \leq \sum_i \delta_i \cdot O(\frac{1}{s}) \leq O(\frac{d(u,v)}{s})$.
- Then

$$E[d^T(u,v)] \leq \sum_{s=1}^{n-2} E[d^T_{w_s}(u,v)] = \sum_{s=1}^{n-2} O\Big(\frac{d(u,v)}{s}\Big) = O(\log n) \cdot d(u,v)$$

Distortion must be $\Omega(\log n)$

Definition (Expander graph)

An undirected graph $G = (V, E)$ is called an (n, d, α)-expander graph if

- $|V| = n$
- constant degree: $deg(v) = d \; \forall v \in V$
- edge expansion

$$\alpha = \min_{1 \leq |S| \leq n/2} \frac{|\delta(S)|}{|S|}$$

- Random d-regular graphs are good expanders w.h.p.
- The diameter of expanders is $\Theta(\log n)$.

Theorem (Bartal '96)

A randomized tree embedding of any (n, d, α)-expander graph (d, α constants) must have an edge with expected distortion of $\Omega(\log n)$.

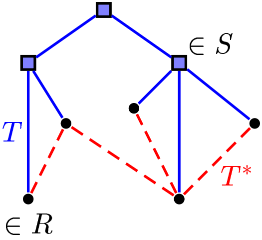
Steiner nodes are not really necessary

Theorem (Gupta '01)

Given a weighted tree $T = (V, E, c)$, where the node set $V = R \dot\cup S$ consists of required vertices R and Steiner nodes S. Then in linear time, one can find a weighted tree $T^* = (R, E^*, c^*)$ such that

$$d^T(u,v) \leq d^{T^*}(u,v) \leq 8 \cdot d^T(u,v)$$

where d^T and d^{T^*} are the induced tree metrices.



Derandomization

Theorem (FRT + Gupta + Charikar et al.)

Given a complete graph $G = (V, E)$ with metric cost function $c : E \to \mathbb{Q}_+$. One can find deterministically, in polynomial time: spanning trees $T_1, \ldots, T_q$ on V, costs $d_i : T_i \to \mathbb{Q}_+$ and probabilities $\lambda_i > 0$, $\lambda_1 + \ldots + \lambda_q = 1$ where $q = poly(n)$. Then

- *For $u, v \in V$ and $i = 1, \ldots, q$ one has $c(u, v) \leq d^{T_i}(u, v)$*
- *For any $u, v \in V$ one has*

$$\sum_{i=1}^{q} \lambda_i \cdot d^{T_i}(u, v) \leq O(\log n) \cdot c(u, v).$$

Here $d^{T_i} : V \times V \to \mathbb{Q}_+$ is the tree metric induced by T_i and d_i.

Part 20
Introduction into Primal dual algorithms

Source: *Approximation Algorithms* (Vazirani, Springer Press)

A generic problem

Situation: We want to approximate a problem, which (in many cases) is of the form

$$\begin{aligned} \min \sum_{j=1}^{n} c_j x_j & & \\ \sum_{j=1}^{n} a_{ij} x_j &\geq& b_i \; \forall i = 1, \ldots, m \\ x_j &\in& \{0,1\} \quad \forall j = 1, \ldots, n \end{aligned}$$

Examples so far: Set Cover, Steiner tree, Vertex Cover,...

A primal-dual pair

Primal "covering" LP:

$$
\begin{aligned}
\min \sum_{j=1}^{n} c_j x_j & & & (P) \\
\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & \quad \forall i = 1, \ldots, m \\
x_j & \geq & 0 & \quad \forall j = 1, \ldots, n
\end{aligned}
$$

Dual "packing" LP:

$$
\begin{aligned}
\max \sum_{i=1}^{m} b_i y_i & & & (D) \\
\sum_{i=1}^{m} a_{ij} y_i & \leq & c_j & \quad \forall j = 1, \ldots, n \\
y_i & \geq & 0 & \quad \forall i = 1, \ldots, m
\end{aligned}
$$

A generic Approximation algorithm

Generic primal-dual algorithm:

(1) $x := \mathbf{0}$, $y = \mathbf{0}$

(2) WHILE x not feasible DO

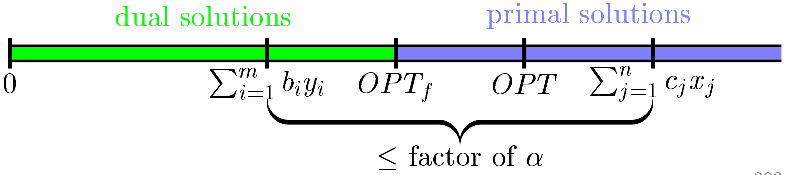
 (3) Increase dual variables in a suitable way until some dual constraint j becomes tight

 (4) Set $x_j := 1$

(5) RETURN x

Generic analysis:

- Show: At the end x is integer and feasible for primal
- Show: At the end y is feasible for dual
- Show: $\sum_{j=1}^{n} c_j x_j \leq \alpha \cdot \sum_{i=1}^{m} b_i y_i$ (α is the apx factor)



Relaxed complementary slackness

Lemma

Let $\alpha, \beta \geq 1$. Let x, y be primal/dual feasible solutions obtained by the algorithm. If

(A) *Relaxed primal compl. slack.:* $x_j > 0 \Rightarrow c_j \leq \alpha \sum_{i=1}^m a_{ij} y_i$

(B) *Relaxed dual compl. slack.:* $y_i > 0 \Rightarrow \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$

Then $APX \leq \alpha \cdot \beta \cdot OPT_f$.

- Let APX be the cost of the produced solution. Then

$$APX = \sum_{j=1}^n c_j x_j \overset{(A)}{\leq} \sum_{j=1}^n x_j \Big(\alpha \sum_{i=1}^m a_{ij} y_i \Big) = \alpha \sum_{i=1}^m y_i \sum_{j=1}^n a_{ij} x_j$$

$$\overset{(B)}{\leq} \alpha\beta \sum_{i=1}^m y_i b_i \overset{y \text{ dual feasible}}{\leq} \alpha\beta \cdot OPT_f \quad \square$$

Part 21
Steiner Forest

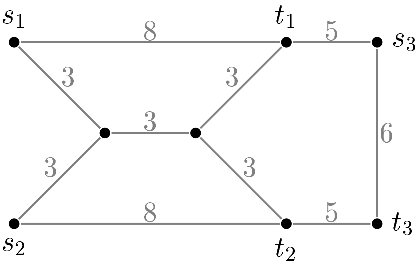
Source: *Approximation Algorithms* (Vazirani, Springer Press)

Steiner Forest

Problem: Steiner Forest

- Given: Undirected graph $G = (V, E)$, edge cost $c : E \to \mathbb{Q}_+$, terminal pairs $(s_1, t_1), \ldots, (s_k, t_k)$
- Find: Minimum cost subgraph F connecting all terminal pairs:

$$OPT = \min_{F \subseteq E} \left\{ \sum_{e \in F} c(e) \mid \forall i = 1, \ldots, k : F \text{ connects } s_i \text{ and } t_i \right\}$$



Steiner Forest

Problem: STEINER FOREST

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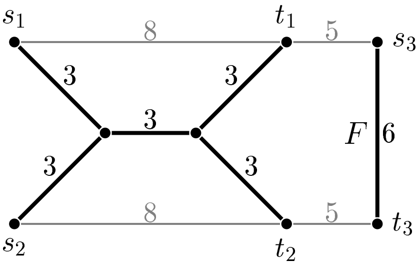


The LP relaxation

- For any $S \subseteq V$ define cut requirement

$$f(S) = \begin{cases} 1 & \text{if } \exists i : |S \cap \{s_i, t_i\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Primal LP relaxation:

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & & & \qquad (P) \\ \sum_{e \in \delta(S)} x_e & \geq & f(S) & \quad \forall S \subseteq V \\ x_e & \geq & 0 & \quad \forall e \in E \end{aligned}$$

Dual LP:

$$\begin{aligned} \max \sum_{S \subseteq V} f(S) y_S & & & \qquad (D) \\ \sum_{S : e \in \delta(S)} y_S & \leq & c_e & \quad \forall e \in E \\ y_S & \geq & 0 & \quad \forall S \subseteq V \end{aligned}$$

Preliminaries

- For $F \subseteq E, S \subseteq V$: $\delta_F(S) = \{\{u,v\} \in F \mid u \in S, v \notin S\}$
- A cut $S \subseteq V$ is violated by $F \subseteq E$, if there is a terminal pair (s_i, t_i) with $|\{s_i, t_i\} \cap S| = 1$ but $\delta_F(S) = \emptyset$
- A cut S is active w.r.t. F, if S is violated and minimal (i.e. there is no subset $S' \subset S$ that is also violated).
- An edge e is tight w.r.t. a dual solution $(y_S)_S$ if $\sum_{S:e\in\delta(S)} y_S = c_e$
  (i.e. if the dual constraint of c_e satisfied with equality).

The algorithm

(1) $F := \emptyset$, $y := \mathbf{0}$

(2) WHILE $\exists$ violated cut DO

 (3) Increase simultaneously y_S for all active cuts S, until some edge e gets tight

 (4) Add the tight edge e to F

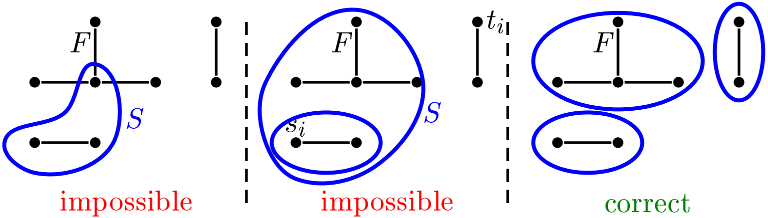
(5) Compute an arbitrary minimal feasible solution $F' \subseteq F$

The active cuts

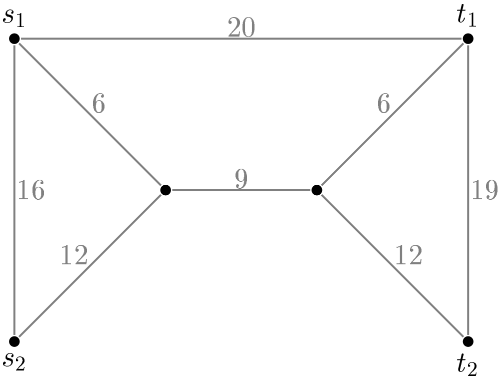
Lemma

The active cuts w.r.t. $F \subseteq E$ are connected components of F.

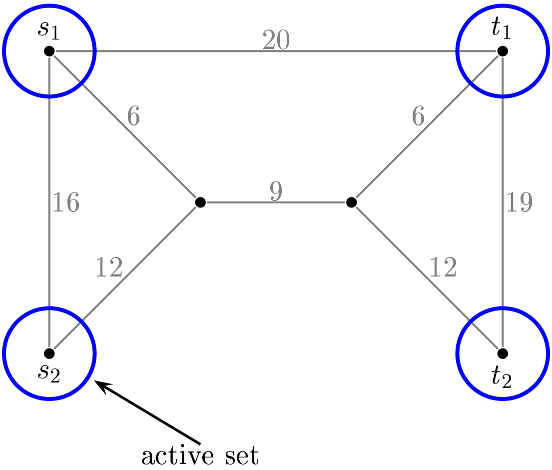
- Consider active cut S (S minimal, $f(S) = 1$, $\delta_F(S) = \emptyset$).
- $\delta_F(S) = \emptyset \Rightarrow$ connected components of F are either fully contained in S or fully outside
- S is violated, hence there is a pair $|\{s_i, t_i\} \cap S| = 1$
- The connected component of F inside S that contains s_i is also violated. Hence, S is a single connected component (or we would have a contradiction). □



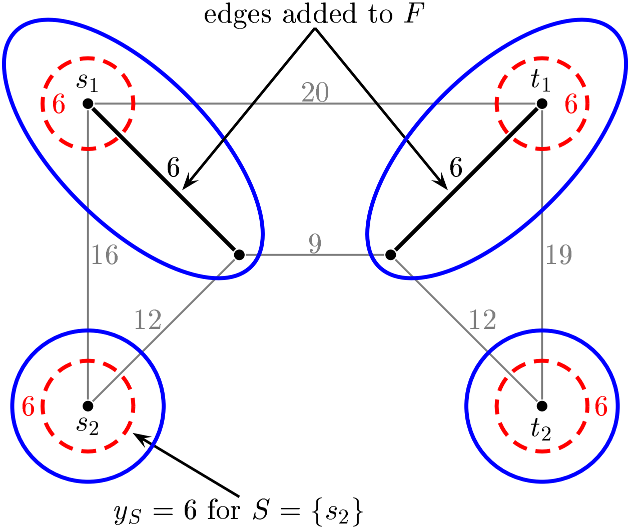
Example



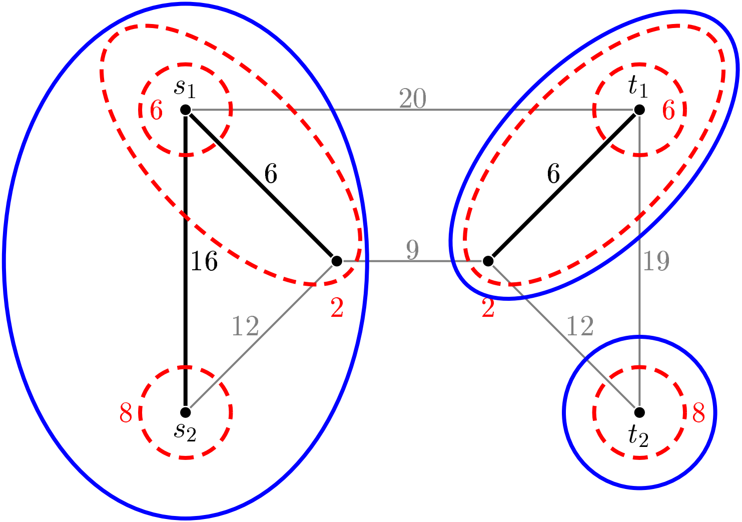
Example



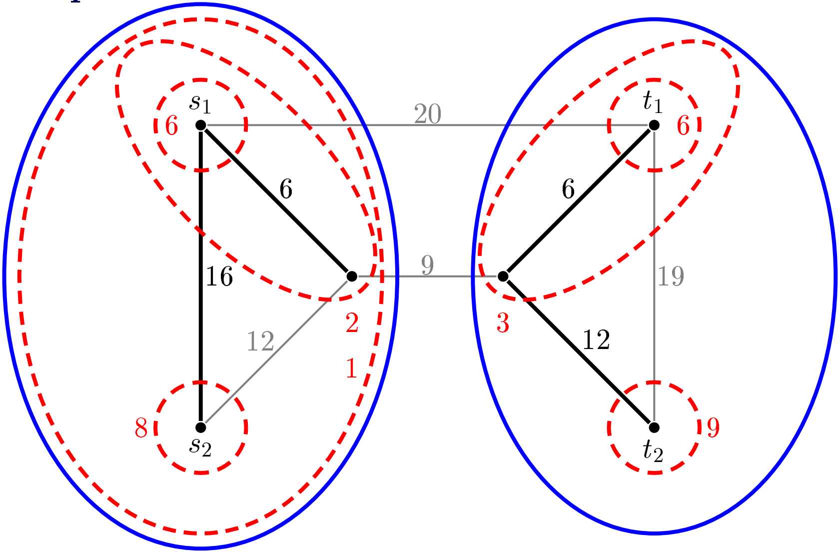
Example



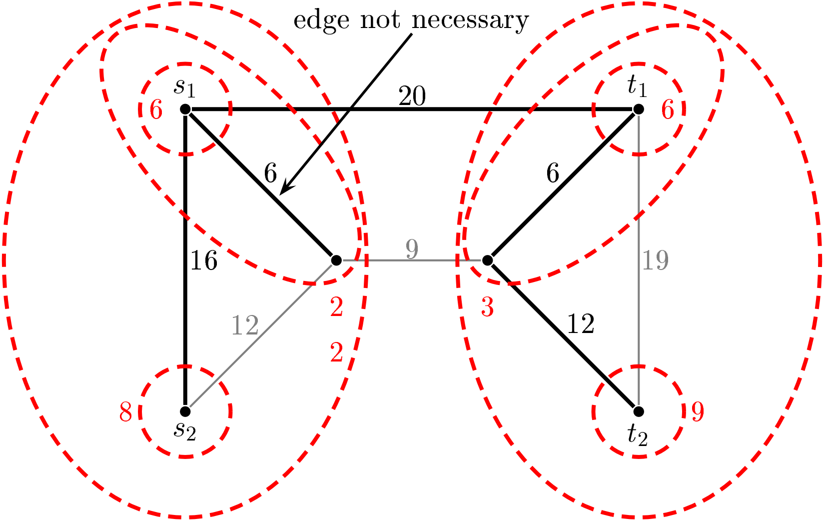
Example



Example

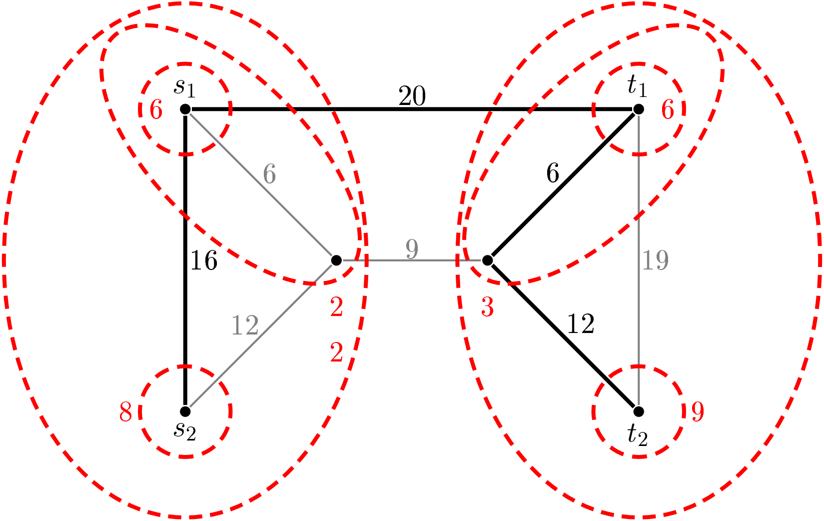


Example



F **at the end of WHILE loop**

Example



Solution F'

Feasibility

Lemma

F' is a feasible solution.

- Let F be the solution at the end of the WHILE loop.
- F is feasible, because there is no violated cut.
- We do not delete necessary edges, hence F' is also feasible. $\square$

Lemma

y is dual feasible, i.e. $\sum_{S:e\in\delta(S)} y_S \leq c_e$ for all $e \in E$.

- Each time that an edge e gets tight (i.e. $\sum_{S:e\in\delta(S)} y_S = c_e$), we add it to F.
- We increase y_S only for violated cuts – not for cuts containing edges of F. $\square$

The main analysis (1)

Lemma

Let y be the dual solution at the end of the algorithm. Then

$$APX = \sum_{e \in F'} c_e \le 2 \sum_{S \subseteq V} y_S \le 2 \cdot OPT_f .$$

$$\sum_{e \in F'} c_e \overset{e \text{ tight}}{=} \sum_{e \in F'} \Bigg(\sum_{S : e \in \delta(S)} y_S \Bigg) = \sum_{S \subseteq V} |\delta_{F'}(S)| \cdot y_S \overset{(*)}{\le} \sum_{S \subseteq V} 2 y_S$$

- Consider any iteration i. Let α be the amount by which the dual variables y_S were increased. We show (*) by proving

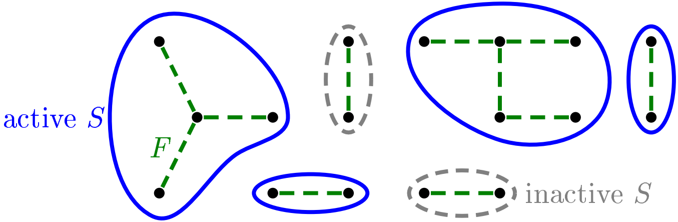
$$\alpha \cdot \sum_{S \text{ active in it.} i} |\delta_{F'}(S)| \le 2 \cdot \alpha \cdot \#\text{active sets in it.} i$$

The main analysis (2)

- Consider an intermediate iteration i with intermediate F.
- **Remark:** $F'\backslash F$ might contain edges that are added later $F\backslash F'$ might contain edges that are deleted at the end.
- Claim:

$$\sum_{S \text{ active in it.} i} |\delta_{F'}(S)| \leq 2 \cdot \#\text{active sets in iteration } i$$

- Shrink connected components of $F \rightarrow H'$ (S becomes node v_S). Nodes v_S steming from active cuts S are active nodes, others are inactive nodes



- H' is a forest. Degrees are preserved.

The main analysis (2)

- Consider an intermediate iteration i with intermediate F.
- **Remark:** $F' \backslash F$ might contain edges that are added later $F \backslash F'$ might contain edges that are deleted at the end.
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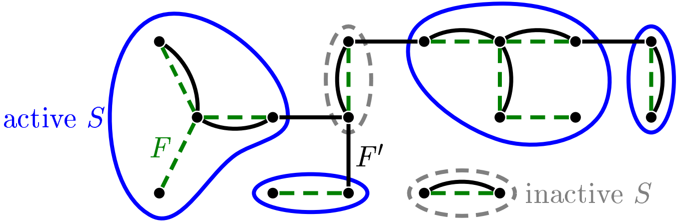

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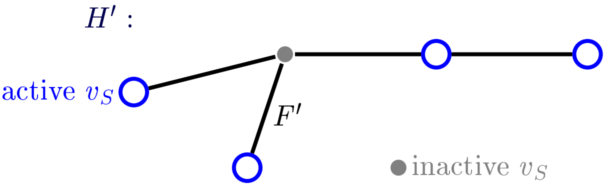

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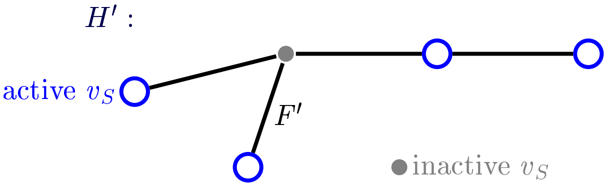
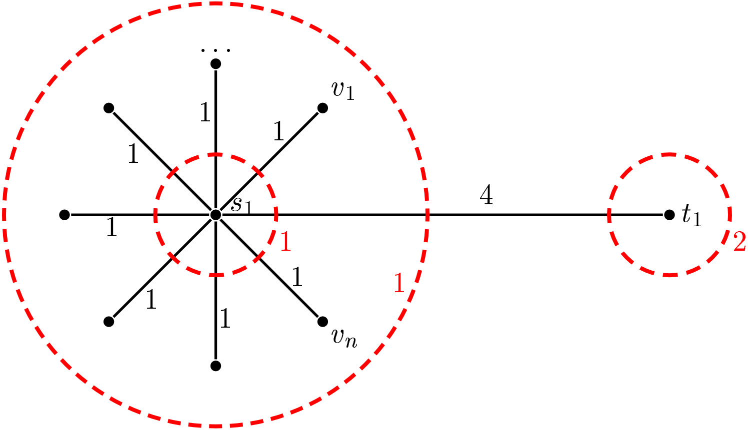
The main analysis (2)



- Consider non-singleton leaf v_S. Edge to v_S was not deleted. Hence $f(S) = 1$. But then S was active (since S is a connected component of F at iteration i).
- Average degree over *all* nodes in a forest is ≤ 2 (since # edges $\leq$ # nodes) and each edge contributes at most 2 to the degrees.
- Inactive nodes are inner nodes of degree ≥ 2, hence average degree of active nodes $\leq$ average degree ≤ 2. □

Deleting redundant edges is crucial



Observation: Without the pruning step at the end of the algorithm, the solution would cost $n + 4$ instead of 4.

Conclusion

Theorem

The primal dual algorithm produces a 2-approximation in time $O(n^2 \log n)$.

Remark: The algorithm works whenever the requirement function $f : 2^V \to \{0, 1\}$ is proper, that means

- $f(V) = 0$
- $f(S) = f(V \backslash S)$ (symmetry)
- If $A, B \subseteq V$ are disjoint and $f(A \cup B) = 1$ then $f(A) = 1$ or $f(B) = 1$.

Note: Function f for Steiner Forest is proper.

State of the art

- There is no $\frac{96}{95}$-approximation algorithm unless $\mathbf{NP} = \mathbf{P}$ (same ratio as for the special case of Steiner tree).
- There is still no better than 2-approximation known.
- The integrality gap of the considered LP is in fact exactly 2.
- There is also no other LP formulation known, which might have a smaller gap.

Part 22
Facility Location

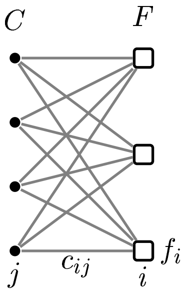
Source: *Approximation Algorithms* (Vazirani, Springer Press)

Facility Location

Problem: Facility Location

- Given: Facilities F, cities C, opening cost f_i for every facility i. Metric cost c_{ij} for connecting city j to facility i.
- Find: Set of facilities I and an assignment $\phi : C \to I$ of cities to opened facilities, minimizing the total cost:

$$OPT = \min_{I \subseteq F, \phi: C \to I} \left\{ \sum_{i \in I} f_i + \sum_{j \in C} c_{\phi(j),j} \right\}$$



- **Remark:** Without the metric assumption, the problem becomes $\Theta(\log n)$-hard.
- We assume w.l.o.g. $c_{ij}, f_i \in \mathbb{Z}_+$

Facility Location

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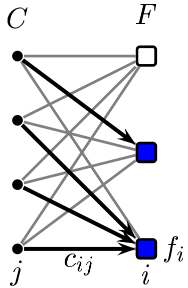


- **Remark:** Without the metric assumption, the problem becomes $\Theta(\log n)$-hard.
- We assume w.l.o.g. $c_{ij}, f_i \in \mathbb{Z}_+$

The primal dual pair

Primal LP:

$$\min \sum_{i,j} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$

$$\begin{array}{rcll} \sum_{i\in F} x_{ij} & \geq & 1 & \forall j \in C \\ x_{ij} & \leq & y_i & \forall i \in F\ \forall j \in C \\ x_{ij} & \geq & 0 & \forall i \in F\ \forall j \in C \\ y_i & \geq & 0 & \forall i \in F \end{array}$$

Dual LP:

$$\max \sum_{j\in C} \alpha_j$$

$$\begin{array}{rcll} \alpha_j & \leq & c_{ij} + \beta_{ij} & \forall i \in F\ \forall j \in C \\ \sum_{j\in C} \beta_{ij} & \leq & f_i & \forall i \in F \\ \alpha_j & \geq & 0 & \forall j \in C \\ \beta_{ij} & \geq & 0 & \forall i \in F\ \forall j \in C \end{array}$$

Intuition:

- α_j is the amount that city j "pays" in total.
- β_{ij} is what city j "pays" to open facility i.

The algorithm - Phase 1:

Initially all cities are unconnected

$\alpha := \mathbf{0}, \beta := \mathbf{0}, F_t := \emptyset$

WHILE not all cities are connected DO

  FOR ALL unconnected cities j DO

    Increase α_j (by 1 per time unit)

    For tight edges $\alpha_j = c_{ij} + \beta_{ij}$ increase also β_{ij}

  IF $\sum_j \beta_{ij} = f_i$ (new) THEN

    open facility i temporarily ($F_t := F_t \cup \{i\}$)

    FOR ALL cities j where edge (i, j) is tight DO

      connect city to facility i

      facility i is connection witness of j: $w(j) := i$

Phase 2:

Let $H = (F_t, E')$ with $(i, i') \in E'$ if $\exists j \in C : \beta_{ij}, \beta_{i'j} > 0$

Open a maximal independent set $I \subseteq F_t$

FOR ALL $j \in C$ DO

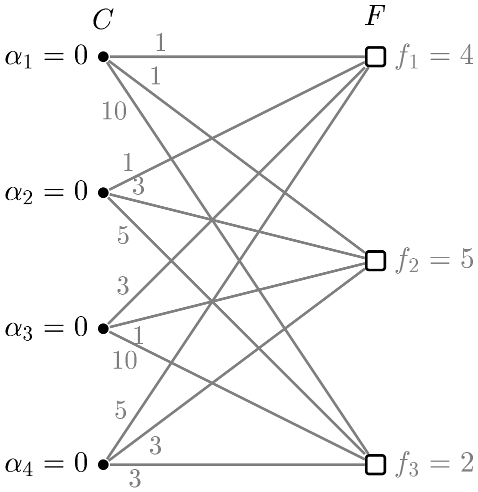
  IF $\exists j \in I : \beta_{ij} > 0$ THEN $\varphi(j) := i$ (j directly conn.)

  ELSE IF $w(j) \in I$ THEN $\varphi(j) := w(j)$ (j directly conn.)

  ELSE $\varphi(j) :=$ a neighbour of $w(j)$ in H (j indir. conn.)

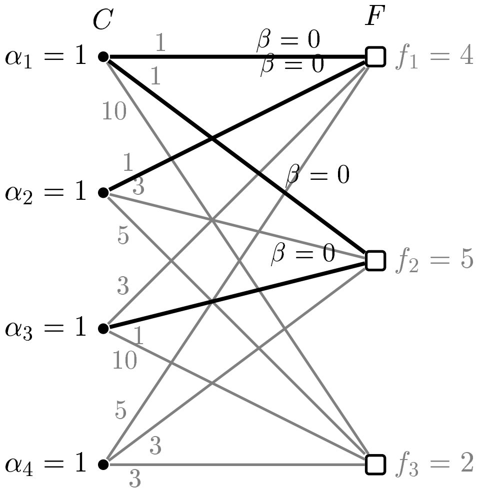
Example:

Phase 1 - Time: 0



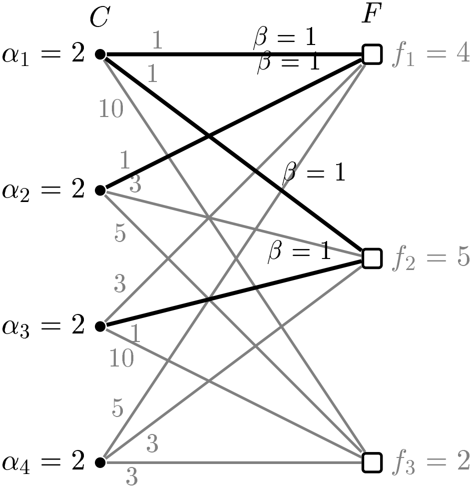
Example:

Phase 1 - Time: 1



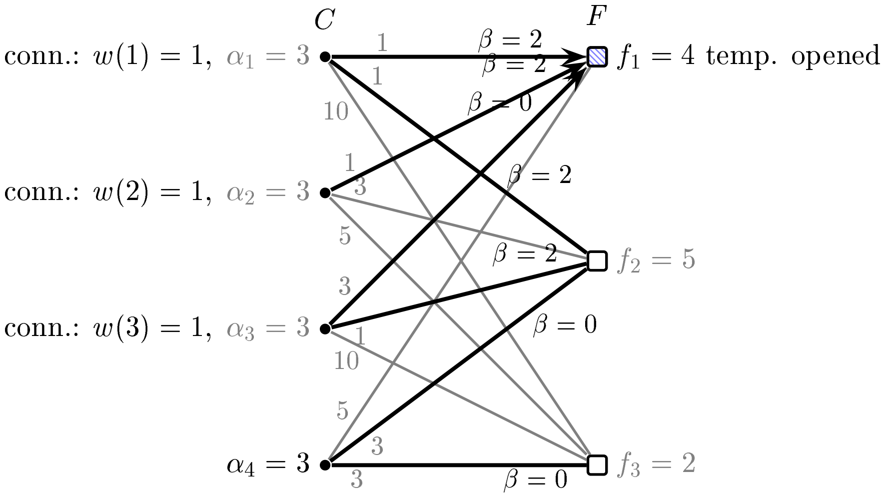
Example:

Phase 1 - Time: 2



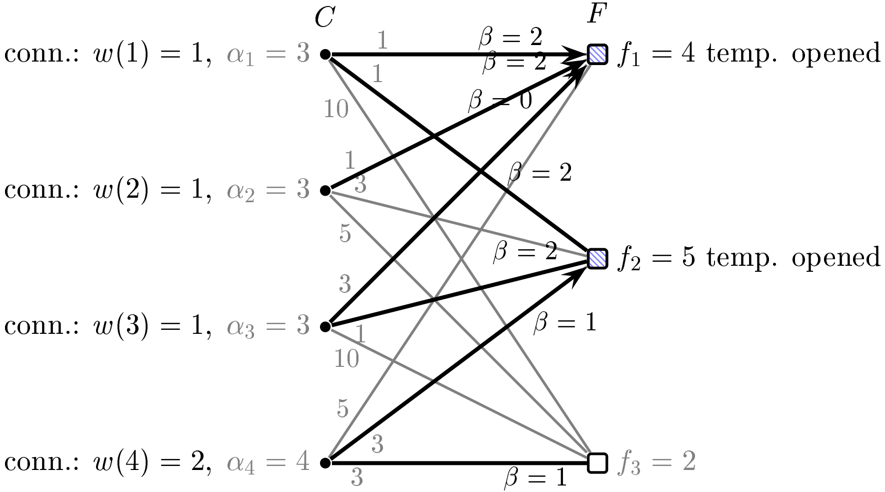
Example:

Phase 1 - Time: 3



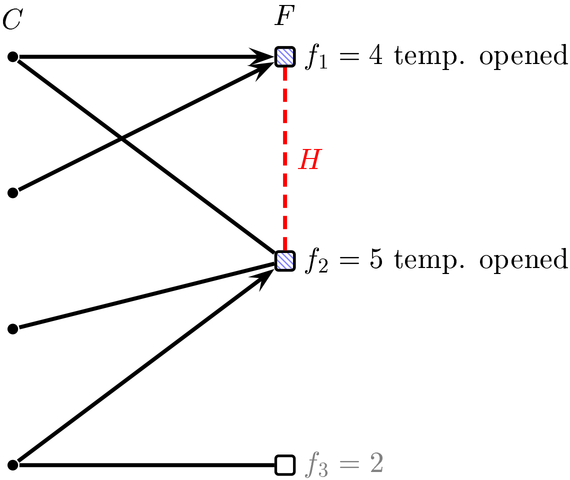
Example:

Phase 1 - Time: 4



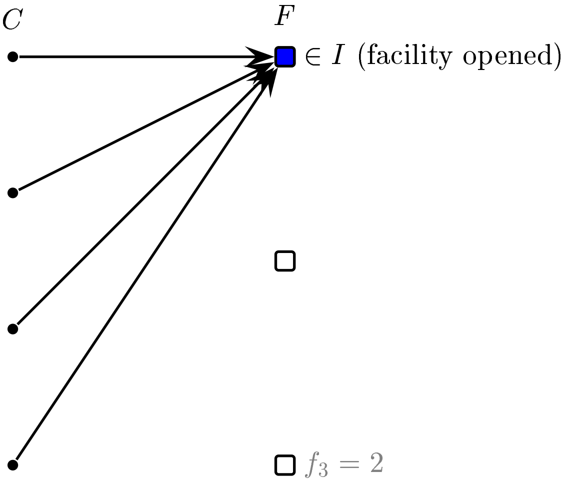
Example:

Phase 2: Graph H



Example:

Phase 2: The solution



Analysis

Theorem

One has $\sum_{j\in C} c_{\varphi(j),j} + \sum_{i\in I} f_i \leq 3\sum_{j\in C}\alpha_j$.

We account the dual "payments"

$$\alpha_j^f := \text{payment for opening} \quad := \begin{cases} \beta_{\varphi(j),j} & \text{if } j \text{ directly connected} \\ 0 & \text{if } j \text{ is indirectly conn.} \end{cases}$$

$$\alpha_j^c := \text{payment for connection} \quad := \begin{cases} c_{\varphi(j),j} & \text{if } j \text{ directly connected} \\ \alpha_j & \text{if } j \text{ is indirectly conn.} \end{cases}$$

Claim: $\alpha_j = \alpha_j^f + \alpha_j^c$.

- For indirectly connected cities: clear
- For directly connected cities: $\alpha_j = c_{\varphi(j),j} + \beta_{\varphi(j),j}$ because edge $(\phi(j), j)$ was tight.

Bounding the opening costs

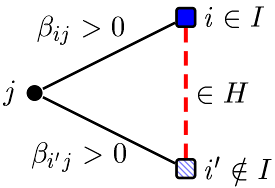
Lemma

The dual prices pay for the opening cost, i.e.

$$\sum_{i\in I} f_i = \sum_{j\in C} \alpha_j^f .$$

- A facility $i \in I$ was temporarily opened because $\sum_j \beta_{ij} = f_i$
- All j with $\beta_{ij} > 0$ must be directly connected to i because: We opened an independent set in H in Phase 2, hence any $i' \in F_t$ with $\beta_{i'j} > 0$ is not in I
- Thus all j with $\beta_{ij} > 0$

$$\sum_{j:\phi(j)=i} \alpha_j^f = \sum_{j:\beta_{ij}>0} \beta_{ij} \overset{i \text{ temp opened}}{=} f_i$$



- The claim follows from

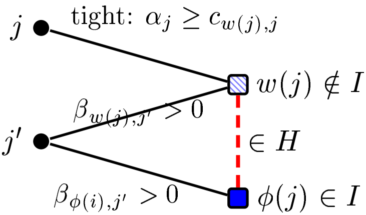
$$\sum_{j\in C} \alpha_j^f = \sum_{i\in I} \sum_{j:\phi(j)=i} \alpha_j^f = \sum_{i\in I} f_i \quad \square$$

Bounding the connection cost

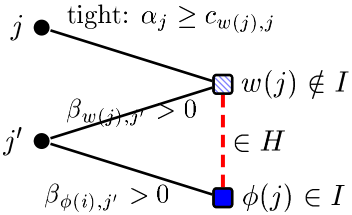
Lemma

For any city $j \in C$ one has $c_{\varphi(j),j} \leq 3\alpha_j^c$.

- If j directly connected, then even $\alpha_j^c = c_{\varphi(j),j}$. Next, suppose j is indirectly connected.
- Then there is an edge $(w(j), \phi(j)) \in H$ (since j was indirectly connected).
- This edge implies that there is a $j' \in C$ with $\beta_{\varphi(j),j'} > 0, \beta_{w(j),j'} > 0$.



Bounding the connection cost (2)



- Event $\beta_{w(j),j} > 0$ only happened if $\alpha_j \geq c_{w(j),j}$. For the same reason: $\alpha_{j'} \geq c_{w(j),j'}$ and $\alpha_{j'} \geq c_{\phi(j),j'}$.
- **Claim $\alpha_j \geq \alpha_{j'}$:** Consider the time t, when $w(j)$ was temporarily opened. Since $w(j)$ is connection witness of j, $\alpha_j \geq t$. At this time t, it was $\beta_{w(j),j'} > 0$ (since if $\beta_{w(j),j'} = 0$ at that time, then $\beta_{w(j),j'} = 0$ forever). At the latest at this time t, also j' was connected and $\alpha_{j'}$ stopped growing. Hence $\alpha_j \geq t \geq \alpha_{j'}$.
- Then

$$c_{\phi(j),j} \overset{\text{metric ineq.}}{\leq} \underbrace{c_{w(j),j}}_{\leq \alpha_j} + \underbrace{c_{w(j),j'}}_{\leq \alpha_{j'} \leq \alpha_j} + \underbrace{c_{\phi(j),j'}}_{\leq \alpha_{j'} \leq \alpha_j} \leq 3\alpha_j = 3\alpha_j^c \quad \square$$

Conclusion

Theorem

The algorithm produces a 3-approximation in time $O(m \cdot \log(m))$, where $m = |C| \cdot |F|$ is the number of edges.

State of the art:

Theorem (Byrka '07)

There is a 1.499-apx for FACILITY LOCATION.

- The integrality gap for the considered LP lies in $[1.463, 1.499]$.

Theorem

There is no polynomial time 1.463-apx for FACILITY LOCATION *unless* $\mathbf{NP} \subseteq \mathbf{DTIME}(n^{O(\log \log n)})$.

Part 23
Insertion: Semidefinite Programming

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Positive definite matrices

Definition (positive semidefinite Matrix)

A matrix $A \in \mathbb{R}^{n\times n}$ is called positive semi-definite if

$$\forall x \in \mathbb{R}^n : x^T A x \geq 0.$$

Theorem (Diagonalization)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric (i.e. $a_{ij} = a_{ji}$), then A is diagonalizable, i.e. one can write

$$A = \underbrace{\begin{pmatrix} \boxed{\begin{matrix}\vdots \\ v_1 \\ \vdots\end{matrix}} & \dots & \boxed{\begin{matrix}\vdots \\ v_n \\ \vdots\end{matrix}} \end{pmatrix}}_{=L} \cdot \underbrace{\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}}_{=D} \cdot \underbrace{\begin{pmatrix} \boxed{\dots \quad v_1 \quad \dots} \\ \vdots \\ \boxed{\dots \quad v_n \quad \dots} \end{pmatrix}}_{=L^T}$$

where $v_i \in \mathbb{R}^n$ is orthonormal Eigenvector for Eigenvalue λ_i, i.e $Av_i = \lambda_i v_i$, $\|v_i\|_2 = 1, v_i^T v_j = 0 \; \forall i \neq j$.

Some useful results

Lemma

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix (v_i orthonormal Eigenvector for λ_i). Then the following statements are equivalent

(1) $\forall x \in \mathbb{R}^n : x^T A x \geq 0$

(2) $\lambda_i \geq 0 \ \forall i$

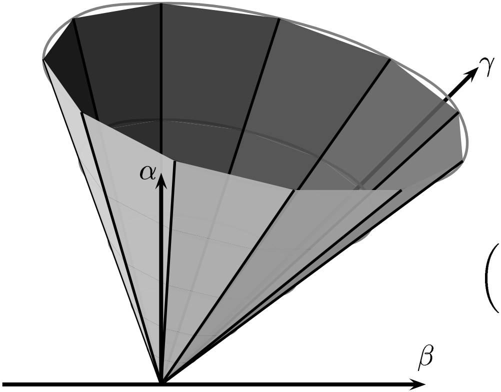
(3) *There is* $W \in \mathbb{R}^{n \times n}$ *with* $A = W^T W$

- (1) $\Rightarrow$ (2). $0 \leq v_i^T A v_i = v_i^T(\lambda_i v_i) = \lambda_i \underbrace{v_i^T v_i}_{=1} = \lambda_i$
- (2) $\Rightarrow$ (3). $A = LDL^T = L\sqrt{D}\sqrt{D}L^T = (\sqrt{D}L^T)^T \underbrace{(\sqrt{D}L^T)}_{=:W}$
- (3) $\Rightarrow$ (1). For any $x \in \mathbb{R}^n$:

$$x^T A x = x^T (W^T W) x = (Wx)^T \cdot (Wx) \geq 0$$

Remark: Matrix W can be found by Cholesky decomposition in $O(n^3)$ arithmetic operations (if $\sqrt{\ }$ counts as 1 operation).

The semidefinite cone



$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$$

- **Def.:** Write $Y \succeq 0$ if Y is positive semidefinite.
- **Fact:** The set

$$\{Y \in \mathbb{R}^{n\times n} \mid Y \succeq 0, Y \text{ symmetric}\} = \text{cone}\{xx^T \mid x \in \mathbb{R}^n\}$$

is a convex, non-polyhedral cone.

A semidefinite program

Given:

- Obj. function vector $C = (c_{ij})_{1 \le i,j \le n} \in \mathbb{Q}^{n \times n}$
- Linear constraints $A_k = (a^k_{ij})_{1 \le i,j \le n} \in \mathbb{Q}^{n \times n}$, $b_k \in \mathbb{Q}$

$$
\begin{aligned}
\max \sum_{i,j} c_{ij} y_{ij} & & \\
\sum_{i,j} a^k_{ij} y_{ij} & \le b_k & \forall k = 1, \dots, m \\
Y & & \text{symmetric} \\
Y & \succeq 0 &
\end{aligned}
$$

- Frobenius inner product: $C \bullet Y := \sum_{i=1}^n \sum_{j=1}^n c_{ij} \cdot y_{ij}$

A semidefinite program

Given:

- Obj. function vector $C = (c_{ij})_{1 \leq i,j \leq n} \in \mathbb{Q}^{n \times n}$
- Linear constraints $A_k = (a^k_{ij})_{1 \leq i,j \leq n} \in \mathbb{Q}^{n \times n}$, $b_k \in \mathbb{Q}$

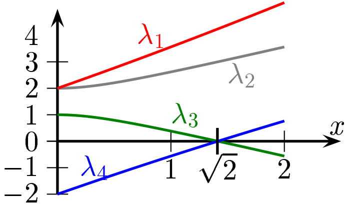
$$
\begin{array}{rcll}
\max C \bullet Y & & & \\
A_k \bullet Y & \leq & b_k & \forall k = 1, \ldots, m \\
Y & & \text{symmetric} & \\
Y & \succeq & 0 &
\end{array}
$$

- Frobenius inner product: $C \bullet Y := \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot y_{ij}$

Pathological situations

- **Case: All solutions might be irrational.** $x = \sqrt{2}$ is the unique solution of

$$\begin{pmatrix} 1 & x & 0 & 0 \\ x & 2 & 0 & 0 \\ 0 & 0 & 2x & 2 \\ 0 & 0 & 2 & x \end{pmatrix} \succeq 0$$



- **Case: All sol. might have exponential encoding length.** Let $Q_1(x) = x_1 - 2, Q_i(x) := \begin{pmatrix} 1 & x_{i-1} \\ x_{i-1} & x_i \end{pmatrix}$. Then

$$Q(x) := \begin{pmatrix} Q_1(x) & 0 & \dots & 0 \\ 0 & Q_2(x) & \dots & 0 \\ \dots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & Q_n(x) \end{pmatrix} \succeq 0$$

if and only if $Q_1(x), \dots, Q_n(x) \succeq 0$. I.e. $x_1 - 2 \geq 0$ and $x_i \geq x_{i-1}^2$, hence $x_n \geq 2^{2^n - 1}$.

Solvability of Semidefinite Programs

Theorem

Given rational input $A_1, \dots, A_m, b_1, \dots, b_m, C, R$ *and* $\varepsilon > 0$, *suppose*

$$SDP = \max\{C \bullet Y \mid A_k \bullet Y \leq b_k \; \forall k; \; Y \, symmetric; \; Y \succeq 0\}$$

is feasible and all feasible points are contained in $B(\mathbf{0}, R)$. *Then one can find a* Y^* *with*

$$A_k \bullet Y^* \leq b_k + \varepsilon, \; Y^* \; symmetric, \; Y^* \succeq 0$$

such that $C \bullet Y^* \geq SDP - \varepsilon$. *The running time is polynomial in the input length,* $\log(R)$ *and* $\log(1/\varepsilon)$ *(in the Turing machine model).*

Solving the separation problem

- **Remark:** We show that we can solve the separation problem, ignore numerical inaccuracies.
- Let infeasible Y be given, we have to find a separating hyperplane.

(1) *Case* $A_k \bullet Y < b_k$*:* return "$A_k \bullet Y \geq b_k$ violated"

(2) *Case* Y *not symmetric:* Find the i, j with $y_{ij} < y_{ji}$. Return "$y_{ij} \geq y_{ji}$ violated".

(3) *Case* Y *not positive semidefinite.* Find eigenvector v with Eigenvalue $\lambda < 0$, i.e. $Yv = \lambda v$. Then

$$\sum_{i,j} v_i^T v_j \cdot y_{ij} = v^T Y v < 0$$

hence return "$\sum_{i,j} v_i^T v_j \cdot y_{ij} \geq 0$ violated".

Vectorprograms

Idea:

$$\begin{aligned} & Y \text{ symmetric and } Y \succeq 0 \\ \Leftrightarrow \quad & \exists W = (v_1, \dots, v_n) \in \mathbb{R}^{n \times n} : W^T W = Y \\ \Leftrightarrow \quad & \exists v_1, \dots, v_n \in \mathbb{R}^n : y_{ij} = v_i^T v_j \end{aligned}$$

SDP:

$$\max \sum_{i,j} c_{ij} y_{ij}$$

$$\begin{aligned} \sum_{i,j} a_{ij}^k \cdot y_{ij} & \leq b_k \quad \forall k \\ Y & \quad \text{sym.} \\ Y & \succeq 0 \end{aligned}$$

Vector program:

$$\max \sum_{i,j} c_{ij} v_i^T v_j$$

$$\begin{aligned} \sum_{i,j} a_{ij}^k \cdot v_i^T v_j & \leq b_k \quad \forall k \\ v_i & \in \mathbb{R}^n \quad \forall i \end{aligned}$$

Observation

The SDP and the vector program are equivalent.

Part 24
MaxCut

Source:

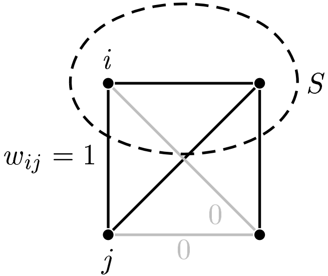
- *Approximation Algorithms* (Vazirani, Springer Press)
- *Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming* (Goemans, Williamson) (link)

Problem definition

Problem: MaxCut

- Given: Complete undirected graph $G = (V, E)$, edge weights $w : E \to \mathbb{Q}_+$
- Find: Cut maximizing the weight of separated edges

$$OPT = \max_{S \subseteq V} \left\{ \sum_{e \in \delta(S)} w(e) \right\}$$



A vector program

- Choose decision variable for any node $i \in V$:

$$v_i = \begin{cases} (1, 0, \dots, 0) & i \in S \\ (-1, 0, \dots, 0) & i \notin S \end{cases}$$

- An exact MaxCut vector program:

$$\max \sum_{(i,j) \in E} \frac{w_{ij}}{2}(1 - v_i^T v_j)$$

$$\begin{array}{rcll} v_i^T v_i & = & 1 & \forall i = 1, \dots, n \\ v_i & \in & \mathbb{R}^n & \forall i = 1, \dots, n \\ v_i & = & (\pm 1, 0, \dots, 0) & \forall i = 1, \dots, n \end{array}$$

- Then

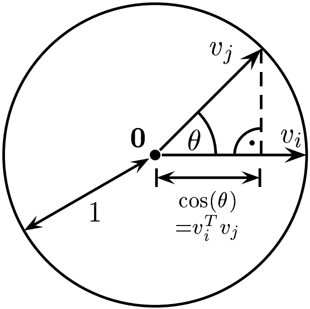
$$\sum_{(i.j) \in E} w_{ij} \cdot \overbrace{\frac{1}{2}(1 - \underbrace{v_i^T v_j}_{\substack{=-1 \text{ if } (i,j) \in \delta(S) \\ +1 \text{ o.w.}}})}^{=1 \text{ if } (i,j) \in \delta(S), \ 0 \text{ o.w.}} = \sum_{(i,j) \in \delta(S)} w_{ij}$$

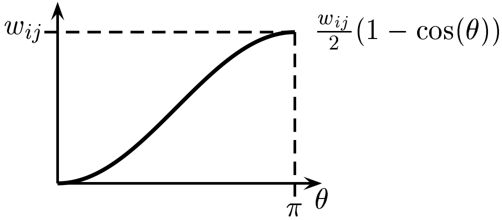
A vector program (2)

The relaxed vector program:

$$\max \sum_{(i,j)\in E} \frac{w_{ij}}{2}(1 - v_i^T v_j)$$

$$\begin{aligned} v_i^T v_i &= 1 \quad \forall i = 1,\dots,n \\ v_i &\in \mathbb{R}^n \quad \forall i = 1,\dots,n \end{aligned}$$



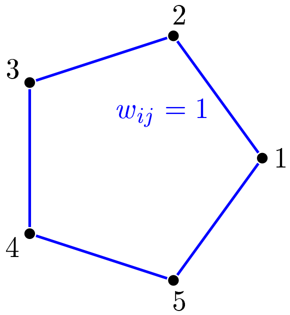


A physical interpretation

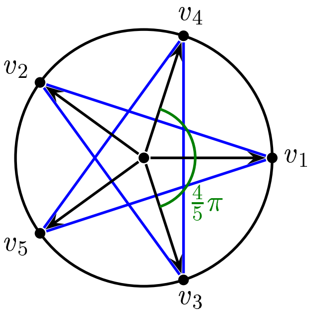
- n vectors on n-dim unit ball.
- Repulsion force of w_{ij} between v_i and v_j

Example:

Graph G



SDP solution:



- $OPT = 4$
- For SDP solution, place $v_1, \ldots, v_5$ equidistantly on 2-dim. subspace. $SDP = 5 \cdot \frac{1}{2}(1 - \cos(\frac{4}{5}\pi)) \approx 4.52$
- Hence integrality gap ≥ 1.13.

The algorithm

Algorithm:

(1) Solve MaxCut vector program $\rightarrow v_1, \ldots, v_n \in \mathbb{Q}^n$
(More precisely: Solve the equivalent SDP, obtain a matrix $Y \in \mathbb{Q}^{n \times n}$. Apply Cholesky decomposition to Y to obtain $v_1, \ldots, v_n$)

(2) Choose randomly a vector r from n-dimensional unit ball

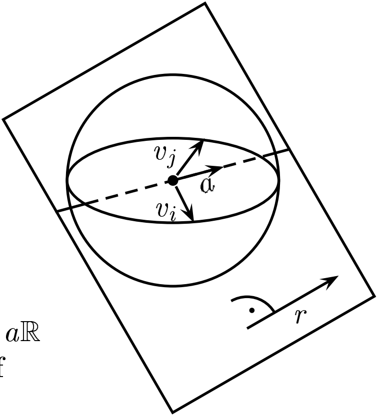
(3) Choose cut $S := \{i \mid v_i \cdot r \geq 0\}$

Theorem

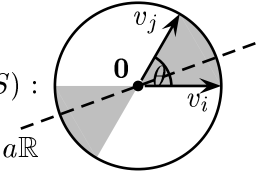
$E[\sum_{(i,j) \in \delta(S)} w_{ij}] \geq 0.87 \cdot OPT$ *(i.e. the algorithm gives an expected* 1.13*-apx).*

Proof

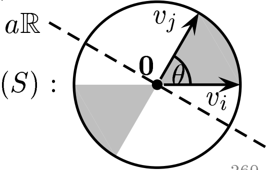
- Consider 2 vectors v_i, v_j with angle $\theta \in [0, \pi]$. Let $\mathbb{R} \cdot a$ be the 1-dim. intersection of the $n-1$-dim. hyperplane $x \cdot r = 0$ with the plane spanned by v_i, v_j
- a has a random direction
- v_i, v_j are separated
  $\Leftrightarrow$ they lie on different sides of line $a\mathbb{R}$
  $\Leftrightarrow$ a lies in one of the 2 gray arcs of angle θ
- $\Pr[v_i \text{ and } v_j \text{ separated}] = 2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{\pi}$
- Expected contribution to APX is $w_{ij} \cdot \frac{\theta}{\pi}$



$(i,j) \in \delta(S)$:



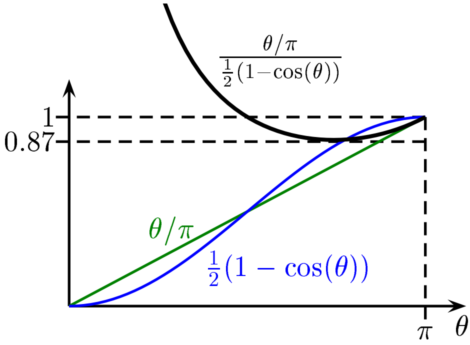
$(i,j) \notin \delta(S)$:



Proof (2)

- Expected contribution of edge (i,j) to APX is $w_{ij} \cdot \frac{\theta}{\pi}$
- Contribution of edge (i,j) to SDP is $w_{ij} \cdot \frac{1}{2}(1 - \cos(\theta))$

$$\frac{E[APX]}{SDP} \geq \min_{0 \leq \theta \leq \pi} \frac{\theta/\pi}{\frac{1}{2}(1-\cos(\theta))} \approx 0.878. \quad \square$$



State of the art

Theorem (Khot, Kindler, Mossel, O'Donnell '05)

There is no polynomial time < 1.138*-approximation algorithm (unless the Unique Games Conjecture is false).*

- That means the presented approximation is the best possible.

Part 25
Max2Sat

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Problem definition

Problem: Max2Sat

- Given: SAT formula $\bigwedge_{C \in \mathcal{C}} C$ on variables $x_1, \ldots, x_n$. Each clause C contains at most 2 literals.
- Find: Truth assignment maximizing the number of satisfied clauses

$$OPT = \max_{a=(a_1,\ldots,a_n)\in\{0,1\}^n} \left|\left\{C \in \mathcal{C} \mid C \text{ true for assignment } a\right\}\right|$$

- **Example:**

$$\underbrace{(\bar{x}_1 \vee x_2)}_{\text{clause}} \wedge (x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (x_1 \vee x_2) \wedge \bar{x}_1$$

  Optimal assignment: $a = (0, 1)$ with 4 satisfied clauses.
- **Remark:** Problem is **NP**-hard though testing wether *all* clauses can be satisfied is easy.

A quadratic program

- **Goal:** Write MAX2SAT as quadratic program

$$\max \sum_{i,j} a_{ij}(1 + y_i y_j) + b_{ij}(1 - y_i y_j)$$
$$y_i^2 = 1$$
$$y_i \in \mathbb{Z}$$

  for suitable coefficients a_{ij}, b_{ij}.
- Here $y_i = 1 \equiv x_i$ true, $y_i = -1 \equiv x_i$ false
- Let $y_0 := 1$ be auxiliary variable.
- Write

$$v(C) = \begin{cases} 1 & \text{if clause } C \text{ true for } y \\ 0 & \text{otherwise} \end{cases}$$

- For clauses with 1 literal

$$v(x_i) = \frac{1 + y_0 y_i}{2}, v(\bar{x}_i) = \frac{1 - y_0 y_i}{2}$$

A quadratic program (2)

- For clause $x_i \vee x_j$

$$\begin{aligned} v(x_i \vee x_j) &= 1 - v(\bar{x}_i) \cdot v(\bar{x}_j) = 1 - \frac{1 - y_0 y_i}{2} \cdot \frac{1 - y_0 y_j}{2} \\ &= \frac{1}{4}(3 + y_0 y_i + y_0 y_j - \overbrace{y_0^2}^{=1} y_i y_j) \\ &= \frac{1 + y_0 y_i}{4} + \frac{1 + y_0 y_j}{4} + \frac{1 - y_i y_j}{4} \end{aligned}$$

- Similar for $\bar{x}_i \vee x_j$ and $\bar{x}_i \vee \bar{x}_j$.
- We obtain promised coefficients a_{ij}, b_{ij} by summing up $\sum_{C \in \mathcal{C}} v(C)$.
- Now: Relax the quadratic program to a (solvable) vector program.

The algorithm

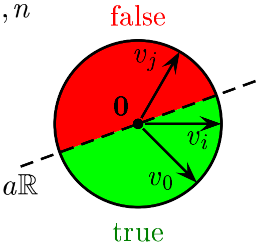
Algorithm:

(1) Solve MaxCut vector program

$$\max \sum_{0 \le i < j \le n} \Big(a_{ij}(1 + v_i v_j) + b_{ij}(1 - v_i v_j) \Big)$$
$$v_i^2 = 1 \ \forall i = 0, \ldots, n$$
$$v_i \in \mathbb{R}^{n+1}$$

(2) Choose randomly a vector r from n-dimensional unit ball

(3) Let $y_i := 1$ for all i that are on the same side of the hyperplane $x \cdot r = 0$ as v_0 (the "truth" vector)



Theorem

Let $APX :=$ #satisfied clauses. Then $E[APX] \ge 0.87 \cdot SDP$.

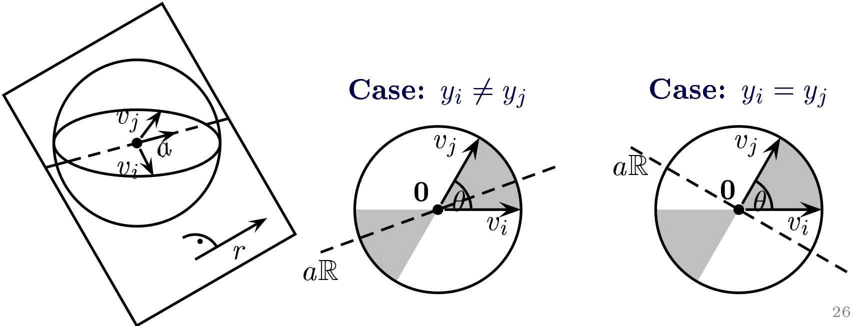
Analysis

Case: Term $b_{ij}(1 - v_i v_j)$ with angle θ between v_i, v_j

- Contribution to $E[APX]$: $2b_{ij} \cdot \Pr[y_i \neq y_j] = 2b_{ij}\frac{\theta}{\pi}$
- Contribution to Vector program: $b_{ij}(1 - \cos(\theta))$
- Gap: $\min_{0 \leq \theta \leq \pi} \frac{2\theta/\pi}{1 - \cos(\theta))} \approx 0.878$

Case: Term $a_{ij}(1 + v_i v_j)$ with angle θ between v_i, v_j

- Contribution to $E[APX]$: $2a_{ij} \cdot \Pr[y_i = y_j] = 2a_{ij}(1 - \frac{\theta}{\pi})$
- Contribution to Vector program: $a_{ij}(1 + \cos(\theta))$
- Gap: $\min_{0 \leq \theta \leq \pi} \frac{2(1 - \theta/\pi)}{1 + \cos(\theta))} \approx 0.878$



State of the art

Theorem (Feige, Goemans '95)

There is a 1.0741*-apx for* MAX2SAT.

Theorem (Lewin, Livnat, Zwick '02)

There is a 1.064*-apx for* MAX2SAT.

Theorem (Hastad '97)

There is no 1.0476*-apx for* MAX2SAT *(unless* **NP** = **P***).*

Theorem (Khot, Kindler, Mossel, O'Donnell '05)

There is no polynomial time 1.063*-apx for* MAX2SAT *(unless the Unique Games Conjecture is false).*

Part 26
Budgeted Spanning Tree

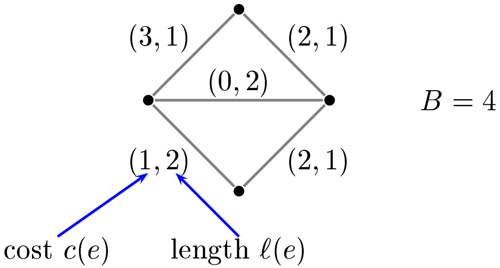
Source: *The Constrained Minimum Spanning Tree Problem* (Goemans, Ravi) (link)

The Budgeted Spanning Tree problem

Problem: Budgeted Spanning Tree

- Given: Undirected graph $G = (V, E)$ with edge costs $c : E \to \mathbb{Q}_+$ and edge lengths $\ell : E \to \mathbb{Q}_+$. Budget B.
- Find: Spanning tree T minimizing the cost, while not exceeding the budget

$$OPT = \max_{\text{spanning tree } T} \left| \left\{ \sum_{e \in T} c_e \mid \sum_{e \in T} \ell_e \leq B \right\} \right|$$

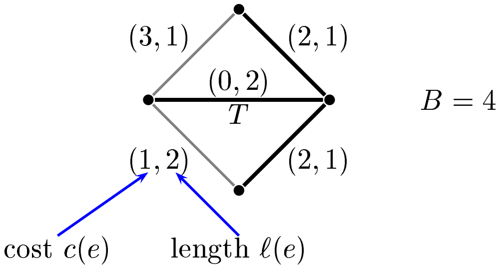


The Budgeted Spanning Tree problem

Problem: BUDGETED SPANNING TREE

- Given: Undirected graph $G = (V, E)$ with edge costs $c : E \to \mathbb{Q}_+$ and edge lengths $\ell : E \to \mathbb{Q}_+$. Budget B.
- Find: Spanning tree T minimizing the cost, while not exceeding the budget

$$OPT = \max_{\text{spanning tree } T} \left| \left\{ \sum_{e \in T} c_e \mid \sum_{e \in T} \ell_e \le B \right\} \right|$$



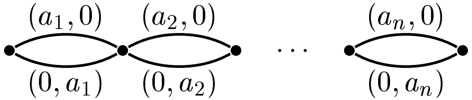
Budgeted Spanning Tree is NP-hard

Recall that Partition is (weakly) **NP**-hard:

Problem: Partition

- Given: Numbers $a_1, \ldots, a_n \in \mathbb{N}$, $S := \sum_{i=1}^{n} a_i$
- Find: $I \subseteq \{1, \ldots, n\} : \sum_{i \in I} a_i = S/2$

Reduction to Budgeted Spanning Tree**:**



- Budget $B := S/2$. There is a feasible tree T of cost $c(T) \leq B, \ell(T) \leq B$ if and only if there is a Partition solution.
- Problem also **NP**-hard for simple graphs (our algorithm will also work for multigraphs).
- Recall: The Spanning Tree problem without a budget is easy.

Lagrangian Relaxation

Original problem:

$$\underbrace{\begin{array}{c}\min_T \; c(T) \\ T \text{ spanning tree} \\ \ell(T) \leq B\end{array}}_{:= OPT}$$

Lagrangian Relaxation:

$$\underbrace{\begin{array}{c}\min_T \; c(T) + z \cdot (\ell(T) - B) \\ T \text{ spanning tree}\end{array}}_{:= OPT_{LR}(z)}$$

Lemma

For any Lagrange multiplier $z \geq 0$: $OPT_{LR}(z) \leq OPT$.

- Let T be the optimum solution: $c(T) = OPT, \ell(T) \leq B$. Then

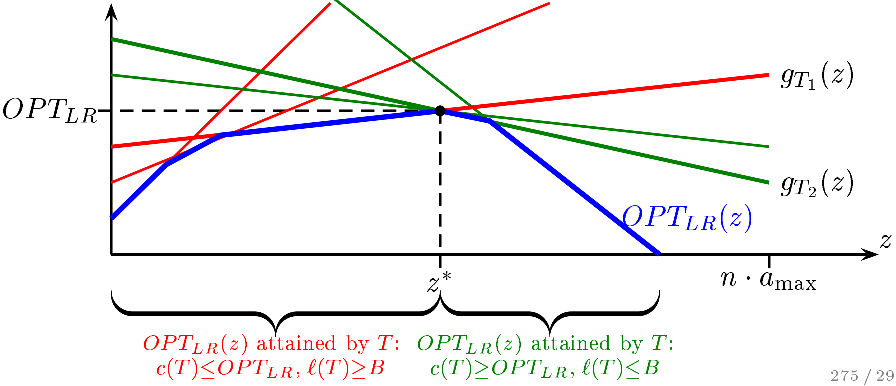
$$OPT = c(T) \geq c(T) + \overbrace{\underbrace{z}_{\geq 0} \cdot \underbrace{(\ell(T) - B)}_{\leq 0}}^{\leq 0} \geq OPT_{LR}(z) \qquad \square$$

Solving the Lagrangian relaxation

Lemma

A sol. z^*, T_1, T_2 can be computed in poly-time where $OPT_{LR} = OPT_{LR}(z^*)$ is attained by T_1, T_2, $\ell(T_1) \geq B \geq \ell(T_2)$.

- Assume w.l.o.g. $c(e), \ell(e) \in \mathbb{Z}$. $a_{\max} := \max\{c(e), \ell(e)\}$
- For any spanning tree T, let $g_T(z) := c(T) + z \cdot (\ell(T) - B)$
- $OPT_{LR}(z) = \min_T\{g_T(z)\}$. Hence $OPT_{LR}(z)$ is concave.



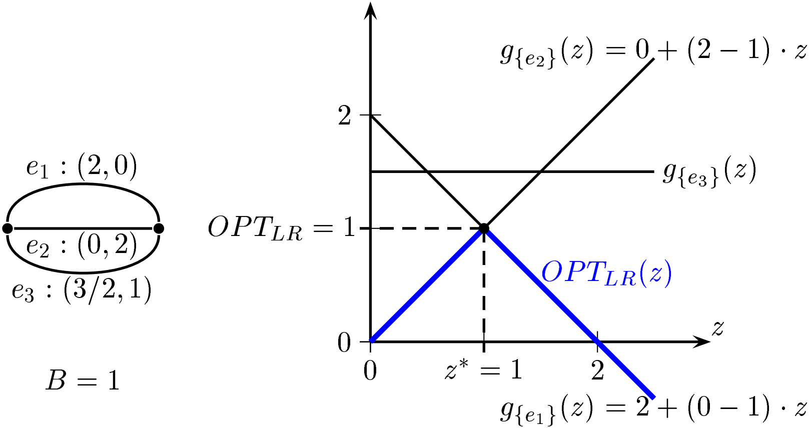
Solving the Lagrangian relaxation (2)

- For a given z, choose $c'(e) := c(e) + z \cdot \ell(e)$, then

$$OPT_{LR}(z) = \min_{\text{sp.tree } T}\{c(T)+z\cdot(\ell(T)-B)\} = \min_{\text{sp.tree } T}\{c'(T)\}-z\cdot B$$

- $OPT_{LR}(0) \geq OPT_{LR}(z)$
- $OPT_{LR}(n \cdot a_{\max}) \leq 0$ (if there is no tree with budget $< B$, then MST w.r.t. $c'(e) := \ell(e) + \frac{1}{n\cdot a_{\max}}c(e)$ is optimal).
- Perform binary search (needs $O(\log(n \cdot a_{\max}))$ iterations):
  (1) $L := 0, R := n \cdot a_{\max}$
  (2) WHILE $|L - R| \geq \frac{1}{4n^2a^2_{\max}}$ DO
    (3) $z := \frac{L+R}{2}$
    (4) $T := MST$ for cost function $c'(e) := c(e) + z \cdot \ell(e)$
    (5) IF $\ell(T) > B$ THEN $L := z$ ELSE $R := z$
  (6) $z^* :=$ rational number in $[L, R]$ with min. denominator
  (7) $T_1 := \text{argmin}_T\{g_T(z^* - \varepsilon)\}$
  (8) $T_2 := \text{argmin}_T\{g_T(z^* + \varepsilon)\}$ ($\varepsilon := \frac{1}{8n^2\cdot a^2_{\max}}$ should suffice) □
- Use: $z^* \in \frac{\mathbb{Z}}{q}$ for some $q \in \{1, \ldots, 4n^2a^2_{\max}\}$

An example

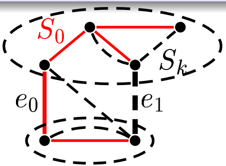


- In this example $OPT = \frac{3}{2}, OPT_{LR} = 1$

Obtaining 2 trees differing in 2 edges

Lemma

One can find opt. Lagrange solutions T_1, T_2 with $\ell(T_1) \geq B, \ell(T_2) \leq B$ which differ in exactly 2 edges.



- Let S_0, S_k the trees returned by the algorithm with $\ell(S_0) \geq B, \ell(S_k) \leq B$ that differ in $|S_k \Delta S_0| := |S_k \backslash S_0| + |S_0 \backslash S_k| = 2k$ edges
- Let $e_0 \in S_0$ be edge maximizing $c'(e) := c(e) + z^* \cdot \ell(e)$. There is an edge $e_1 \in S_k \backslash S_0$ such that $S_1 := S_0 \backslash \{e_0\} \cup \{e_1\}$ is a spanning tree. Since $c'(S_0) = c'(S_k)$, $c'(e_0) \geq c'(e_1)$. On the other hand $c'(S_1) \geq c'(S_0)$ since S_0 has minimal c'-cost. Hence $c'(S_1) = c'(S_0)$ and $|S_1 \Delta S_0| = 2(k-1)$.
- We iterate this to obtain $S_0, \dots, S_k$ with $c'(S_0) = c'(S_1) = \dots = c'(S_k)$ and $|S_i \Delta S_{i+1}| = 2 \; \forall i$.
- Since $\ell(S_0) \geq B$, $\ell(S_k) \leq B$ there must be a pair $(T_1, T_2) := (S_i, S_{i+1})$ with $\ell(S_i) \geq B, \ell(S_{i+1}) \leq B$. □

T_2 is not that bad

Lemma

Let z^*, T_1, T_2 be opt. Lagrange solutions, $\ell(T_1) \geq B, \ell(T_2) \leq B$ s.t. $|T_1 \Delta T_2| = 2$. Then $c(T_2) \leq OPT + c_{\max}$.

- Recall that

$$c(T_1) \leq c(T_1) + \overbrace{z^* \cdot \underbrace{(\ell(T_1) - B)}_{\geq 0}}^{\geq 0} = OPT_{LR}(z^*) \leq OPT$$

- Let e_1, e_2 be edges with $T_2 = (T_1 \backslash \{e_1\}) \cup \{e_2\}$. Then

$$c(T_2) = \underbrace{c(T_1)}_{\leq OPT} - \underbrace{c(e_1)}_{\geq 0} + \underbrace{c(e_2)}_{\leq c_{\max}} \leq OPT + c_{\max} \quad \square$$

A PTAS

Lemma

There is a PTAS for BUDGETED SPANNING TREE.

- Guess the $1/\varepsilon$ many edges of maximum cost in the optimum solution.
- Contract them. Now $c_{\max} \leq \varepsilon \cdot OPT$ in the remaining instance.

State of the art:

- It is not know, whether there is an FPTAS for BUDGETED SPANNING TREE.
- [Hong et al.] can find a tree T with $c(T) \leq (1+\varepsilon)OPT, \ell(T) \leq (1+\varepsilon)B$ in $poly(n, 1/\varepsilon)$ (i.e. a bicriteria FPTAS).

PART 27
k-MEDIAN

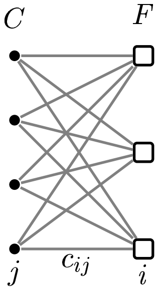
SOURCE: *Approximation Algorithms* (Vazirani, Springer Press)

k-Median

Problem: k-Median

- Given: Facilities F, cities C, parameter $k \in \mathbb{N}$. Metric cost c_{ij} for connecting city j to facility i.
- Find: Set of at most k facilities I and an assignment $\phi : C \to I$ of cities to opened facilities, minimizing the connection cost:

$$OPT := \min_{I \subseteq F, |I| \leq k, \phi : C \to I} \sum_{i \in I} c_{\phi(j),i}$$



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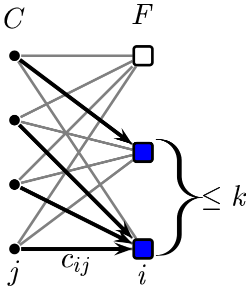


Integer program :

$$\min \sum_{i\in F}\sum_{j\in C} x_{ij}c_{ij}$$

$$\begin{aligned} \sum_{i\in F} x_{ij} &= 1 && \forall j\in C \\ x_{ij} &\le y_i && \forall i\in F\ \forall j\in C \\ \sum_{i\in F} y_i &\le k \\ y_i, x_{ij} &\in \{0,1\} && \forall i\in F\ \forall j\in C \end{aligned}$$

$= OPT$

$\geq$

Lagrangian Relaxation $(\mathbf{z} \geq \mathbf{0})$:

$$\min \sum_{i\in F}\sum_{j\in C} x_{ij}c_{ij} + z\cdot\Big(\sum_{i\in F} y_i - k\Big)$$

$$\begin{aligned} \sum_{i\in F} x_{ij} &= 1 && \forall j\in C \\ x_{ij} &\le y_i && \forall i\in F\ \forall j\in C \\ y_i, x_{ij} &\in \{0,1\} && \forall i\in F\ \forall j\in C \end{aligned}$$

$=: OPT_{LR}(z)$

$=$ optimum facility location value for instance with $f_i := z$ $-zk$

$=: OPT_{FL}(z)$

Approximating the Lagrangean Relaxation (1)

Recall the previous result:

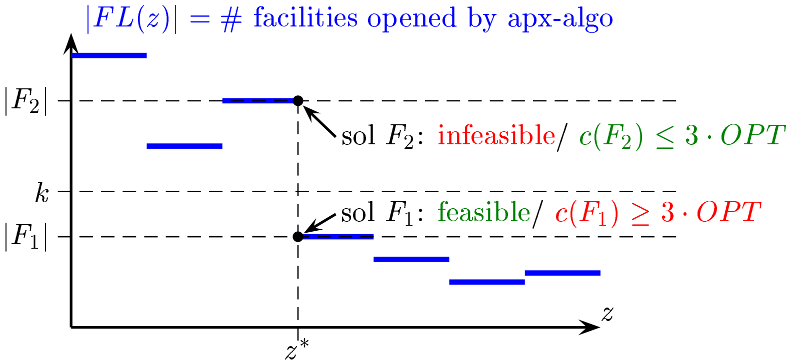
Theorem

One can compute a FACILITY LOCATION *solution in poly-time, with*

$$\textit{connection cost} + 3 \cdot \textit{facility cost} \leq 3 \cdot OPT_{FL}.$$

- Let $FL(z) \subseteq F$ be the set of facilities, opened by approximation algorithm if $f_i := z$ for all facilities $i \in F$.
- For $F' \subseteq F$ and $j \in C$ let $c(F', j) := \min_{i \in F'}\{c_{ij}\}$ be the distance of city j to nearest facility in F'
- Let $c(F') := \sum_{j \in C} c(F', j)$ be the connection cost of a FACILITY LOCATION or k-MEDIAN solution F'.

Approximating the Lagrangean Relaxation (2)



- $|FL(0)| = |F| \geq k, \; \lim_{z\to\infty} |FL(z)| = 1 \leq k$
- By binary search in the interval $[0, |C| \cdot \max_{i,j}\{c_{ij}\}]$, find $z^* \geq 0$, where $|FL(z^*)| \geq k \geq |FL(z^* + \varepsilon)|$
- Let $F_1 := FL(z^* + \varepsilon), \; F_2 := FL(z^*)$ be the obtained approximate solutions (we ignore the ε-term from now on, since it can be made exponentially small).

Bounding the cost of F_1, F_2

Lemma

Choose $0 \leq \lambda \leq 1$ *with* $\lambda|F_1| + (1-\lambda)|F_2| = k$. *Then*

$$\lambda \cdot c(F_1) + (1-\lambda) \cdot c(F_2) \leq 3 \cdot OPT.$$

- Since we use a $(3,1)$-apx algo for Facility Location:

$$\begin{aligned} c(F_1) + 3z \cdot |F_1| &\leq 3 \cdot OPT_{FL}(z) \\ c(F_2) + 3z \cdot |F_2| &\leq 3 \cdot OPT_{FL}(z) \end{aligned}$$

- Adding both inequalities with coefficient λ and $1-\lambda$, resp.:

$$\lambda c(F_1) + (1-\lambda)c(F_2) + 3z \cdot \underbrace{(\lambda|F_1| + (1-\lambda)|F_2|)}_{=k}$$

$$\leq \quad 3 \cdot OPT_{FL}(z) = 3 \cdot OPT_{LR}(z) + 3z \cdot k$$

- The $3zk$ term cancels out and

$$\lambda c(F_1) + (1-\lambda)c(F_2) \leq 3 \cdot OPT_{LR}(z) \leq 3 \cdot OPT \quad \square$$

Combining F_1 and F_2 (1)

Lemma

We can randomly choose a subset $I \subseteq F_1 \cup F_2$ of size $|I| \le k$ of cost $E[c(I)] \le 6 \cdot OPT$.

- We want to choose I s.t.

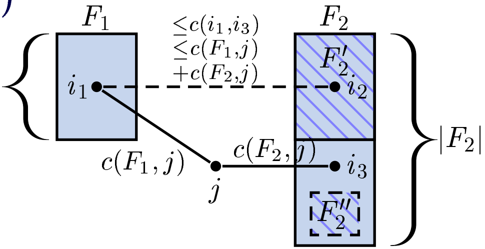
$$E[c(I,j)] \le 2 \cdot \big(\lambda \cdot c(F_1,j) + (1-\lambda) \cdot c(F_2,j)\big).$$

Then

$$\begin{aligned} E[c(I)] &= \sum_{j \in C} E[c(I,j)] \le \sum_{j \in C} 2\big(\lambda \cdot c(F_1,j) + (1-\lambda) \cdot c(F_2,j)\big) \\ &\le 2 \cdot \underbrace{(\lambda \cdot c(F_1) + (1-\lambda) \cdot c(F_2))}_{\le 3 \cdot OPT} \le 6 \cdot OPT \end{aligned}$$

Combining F_1 and F_2 (2)

Case (1): With prob $1 - \lambda$:



- Choose $F_2' \subseteq F_2$ with $|F_2'| = |F_1|$ so that for any facility $i_1 \in F_1$, also the facility $i_2 \in F_2$ minimizing c_{i_1,i_2} is in F_2'
- Choose $F_2'' \subseteq F_2 \backslash F_2'$ with $|F_2''| = k - |F_1|$ uniformly at random. Open $I := F_2' \cup F_2''$.
- Let $i_1 \in F_1$ and $i_3 \in F_2$ be nearest facilities to j. Suppose $i_3 \notin F_2'$ (other case later).
- Note that $\Pr[i_3 \in I] = \frac{k-|F_1|}{|F_2|-|F_1|} = 1 - \lambda$. Hence

$$E[c(I,j)] \le \underbrace{\Pr[i_3 \in I]}_{=1-\lambda} \cdot \underbrace{c(i_3,j)}_{\le c(F_2,j)} + \underbrace{\Pr[i_3 \notin I]}_{=\lambda} \cdot \underbrace{c(i_2,j)}_{\le 2c(F_1,j)+c(F_2,j)}$$

$$\le (1 - \lambda + \lambda) \cdot c(F_2, j) + 2\lambda \cdot c(F_1, j)$$

$$\le c(F_2, j) + 2\lambda \cdot c(F_1, j)$$

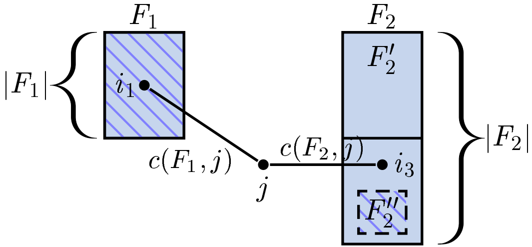
Combining F_1 and F_2 (2)

Case (2): With prob λ:

- Choose $I := F_1 \cup F_2''$
- Then

$$E[c(I,j)] \leq \underbrace{\Pr[i_3 \in I]}_{=1-\lambda} \cdot \underbrace{c(i_3,j)}_{\leq c(F_2,j)} + \underbrace{\Pr[i_3 \notin I]}_{=\lambda} \cdot \underbrace{c(i_1,j)}_{\leq c(F_1,j)}$$

$$\leq \lambda c(F_1,j) + (1-\lambda) c(F_2,j)$$



Combining F_1 and F_2 (3)

- Overall:

$$
\begin{aligned}
& E[c(I,j)] \\
\leq\; & \underbrace{\Pr[\text{case } (1)]}_{=1-\lambda} \cdot \underbrace{E[c(I,j) \text{ in } (1)]}_{2\lambda c(F_1,j)+c(F_2,j)} + \underbrace{\Pr[\text{case } (2)]}_{=\lambda} \cdot \underbrace{E[c(I,j) \text{ in } (2)]}_{\leq \lambda c(F_1,j)+(1-\lambda)c(F_2,j)} \\
\leq\; & \lambda \cdot \underbrace{(\lambda + 2(1-\lambda))}_{\leq 2} \cdot c(F_1,j) + (1-\lambda) \cdot \underbrace{(1+\lambda)}_{\leq 2} \cdot c(F_2,j) \qquad \square
\end{aligned}
$$

- (For case $i_3 \in F_2'$: $E[c(I,j)] \leq \lambda c(F_1,j) + (1-\lambda)c(F_2,j)$).

The main result

Theorem

*There is an expected 6-approximation for k-*MEDIAN *in polynomial time (which can be easily derandomized).*

State of the art:

Theorem (Arya et al.)

One can obtain a $(3 + \varepsilon)$-apx in time $O(n^{2/\varepsilon})$.

- Algorithm uses local search.
- The natural LP relaxation has an integrality gap of 3, but no algorithm is known that achieves this value.