École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 16, 2018

Discrete Optimization (Spring 2018)

Assignment 4

Problem 6 can be **submitted** until March 23 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Consider the polyhedron:

$$P = \begin{cases} x_1 + 2x_2 + x_3 & \leq & 5 \\ 3x_1 + x_2 + x_3 & \leq & 3 \\ x_1 & \leq & 1 \\ x_1 + x_2 & \leq & 2 \\ x_2 + x_3 & \leq & 3 \\ x_1 & \geq & 0 \\ x_1 + x_2 & \geq & 0 \\ x_2 + x_3 & \geq & 0 \end{cases}$$

State which of the following points are vertices of P: $p_0 = (0,0,3)$, $p_1 = (0,1,1), p_2 = (1,4,-4)$, $p_3 = (1/2, 3/2, 0), p_4 = (1,-1,1)$.

Solution:

For each point p, we need to check whether the submatrix of the inequalities that p satisfies with equality has full rank (i.e. equal to 3), and whether p is in P. Proceeding this way, we see that only p_0 and p_4 are vertices.

Problem 2

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of A.

i) Show that $\text{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n \colon A^{-1}x \geq 0\}$.

ii) Show that $\text{cone}(\{a_1, \ldots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{x \in \mathbb{R}^n \colon a_i^{-1}x \geq 0, i = 1, \ldots, k,\, a_i^{-1}x = 0, i = k+1, \ldots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Solution:

i) We obtain the following (where $[n]$ denotes the set $\{1, 2, \ldots, n\}$):

$$\begin{aligned} \text{cone}(\{a_1, \ldots, a_n\}) &= \{x = \sum_{i \in [n]} \lambda_i a_i :\ \lambda_i \in \mathbb{R}_{\geq 0}\ \forall i \in [n]\} = \{x = A\lambda :\ \lambda \in \mathbb{R}^n_{\geq 0}\} = \\ &= \{x \in \mathbb{R}^n :\ A^{-1}x = \lambda,\ \lambda \geq 0\} = \{x \in \mathbb{R}^n \colon A^{-1}x \geq 0\}. \end{aligned}$$

ii) Analogously one has:

$$\begin{aligned} \text{cone}(\{a_1, \ldots, a_k\}) &= \{x = A\lambda :\ \lambda \in \mathbb{R}^n_{\geq 0},\ \lambda_i = 0 \text{ for } i > k\} = \\ &= \{x \in \mathbb{R}^n :\ A^{-1}x = \lambda,\ \lambda \geq 0,\ \lambda_i = 0 \text{ for } i > k\} = \\ &= \{x \in \mathbb{R}^n \colon a_i^{-1}x \geq 0, i = 1, \ldots, k,\, a_i^{-1}x = 0, i = k+1, \ldots, n\}. \end{aligned}$$

Problem 3

Prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geq 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geq 0$. *Hint: Use the version of Farkas' lemma in the lecture notes, Theorem 3.11*

Solution:

The system $Ax \leq b, x \in \mathbb{R}^n$ is feasible if and only if the system $A(x^+ - x^-) + s = b$ has a solution $\bar{x} = [x^+ \ x^- \ s]^T \geq 0$, where $x^+, x^- \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$. We could rewrite the latter system as $\bar{A}\bar{x} = b$ with $\bar{x} \geq 0$, where $\bar{A} = [A \ {-A} \ I_m]$. By applying the Farkas' lemma seen in class to this new system we obtain that the original system $Ax \leq b$ is feasible if and only if for all $\lambda \in \mathbb{R}^m$ such that $\lambda^T[A \ {-A} \ I_m] \geq 0 \iff (\lambda^T A \geq 0, \lambda^T(-A) \geq 0) \iff \lambda^T A = 0$ and $\lambda^T I_m = \lambda^T \geq 0$ one has $\lambda^T b \geq 0$.

Problem 4

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 5 \\ 31 \end{pmatrix}$$

is a conic combination of the x_i.

Write v as a conic combination using only three vectors of the x_i.

Hint: Recall the proof of Carathéodory's theorem

Solution:

We notice that: $4x_1 - 5x_3 - x_4 = 0$, hence we can write

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 - \epsilon(4x_1 - 5x_3 - x_4) = (1 - 4\epsilon)x_1 + 3x_2 + (2 + 5\epsilon)x_3 + (1 - \epsilon)x_4 + 3x_5.$$

We choose $\epsilon = 1/4$ to obtain:

$$v = 3x_2 + \frac{13}{4}x_3 + \frac{5}{4}x_4 + 3x_5$$

We now repeat the procedure, using $x_2 + x_3 - x_4 - x_5 = 0$, and finally we get:

$$v = \frac{1}{4}x_2 + \frac{17}{4}x_4 + 6x_5$$

Problem 5

Consider the following classification problem: we are given $p_1, \ldots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is an hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such α. Show how to solve this problem using linear programming.

Solution:

Consider the following linear program (notice that $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ are variables):

$$\begin{array}{rrcll} \max & \epsilon & & & \\ & ap_i & \geq & 1 + \epsilon & \forall\, p_i \text{ blue} \\ & ap_i & \leq & 1 & \forall\, p_i \text{ red} \\ & \epsilon & \geq & 0 & \end{array}$$

If a separating hyperplane exists, then (by changing the sign of the coefficients or by slightly translating it) we can write it as $ax = 1$ and if $\epsilon = \min\{ap_i : p_i \text{ is blue}\}$, we have that (a, ϵ) is a feasible solution to the linear program with positive objective value. On the other hand, if there is a feasible solution with positive objective value, then the corresponding hyperplane $ax = 1$ strictly separates the blue points from the red.

Problem 6 ($\star$)

Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex.
Hint: Use Problem 2 and Carathéodory's theorem: Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \text{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.

Solution:
Denote with A_X the matrix whose columns are vectors in X, and analogously with $A_{\widetilde{X}}$ the one corresponding to a set $\widetilde{X}$. Points $u, v \in \text{cone}(X)$ can then be written as $u = A_X \lambda_u$ and $v = A_X \lambda_v$ for some vectors $\lambda_u, \lambda_v \geq 0$. Furthermore point $p = \gamma u + (1 - \gamma)v$ for some $\gamma \in [0, 1]$ can be written as $p = \gamma A_X \lambda_u + (1 - \gamma)A_X \lambda_v = A_X(\gamma \lambda_u + (1 - \gamma)\lambda_v)$ so $p \in \text{cone}(X)$ since the vector $\gamma \lambda_u + (1 - \gamma)\lambda_v \geq 0$. This proves that $\text{cone}(X)$ is a convex set.
In order to see that $\text{cone}(X)$ is closed we need to show that for every convergent sequence $(y_k)_{k \in \mathbb{N}}$ where $y_k \in \text{cone}(X)$ we have that $y = \lim_{k \to \infty} y_k$ belongs to $\text{cone}(X)$. By Carathéodory's theorem we know that for every y_k there is a set $\widetilde{X}_k \subseteq X$ of at most n linearly independent vectors such that $y_k \in \text{cone}(\widetilde{X}_k)$. Since there are only finitely many such subsets $\widetilde{X}$, there is one of them such that $\widetilde{X} = \widetilde{X}_k$ for infinitely many k, hence we can restrict our sequence only to those values of k, which we denote by $k_1, k_2, \ldots$. The restricted subsequence $(y_{k_i})_{i \in \mathbb{N}}$ satisfies $y_{k_i} \in \text{cone}(\widetilde{X})$ for every i and has the same limit y. We now claim that $y \in \text{cone}(\widetilde{X})$, which concludes the proof as $\text{cone}(\widetilde{X}) \subseteq \text{cone}(X)$. Notice that this is equivalent to showing that $\text{cone}(\widetilde{X})$ is closed. Let $k = |\widetilde{X}| \leq n$, and let A be a non-singular matrix formed by $A_{\widetilde{X}}$ (whose columns are linearly independent) and $n - k$ other columns. Applying Problem 1.ii to A, it follows that $\text{cone}(\widetilde{X}) = P_k$, which is closed as it is intersection of half-spaces (which are closed sets).

Alternative proof that $\text{cone}(X)$ *is closed:* We claim that

$$\text{cone}(X) = \bigcup_{\substack{\widetilde{X} \subseteq X \\ \widetilde{X} \text{ lin.ind.}}} \text{cone}(\widetilde{X}).$$

By the previous exercise we have that if the vectors in $\widetilde{X}$ are linearly independent, $\text{cone}(\widetilde{X})$ is a polyhedron and thus it is closed (any polyhedron is the intersection of finitely many half spaces, which are closed sets). Since X is a finite set the number of subsets of X is also finite and thus $\bigcup_{\substack{\widetilde{X} \subseteq X \\ \widetilde{X} \text{ lin.ind.}}} \text{cone}(\widetilde{X})$ is a finite union of closed sets, hence it is closed.
We now prove the claim. The "$\supseteq$" direction trivially follows from $\widetilde{X} \subseteq X$ and the conic hull definition. In order to prove "$\supseteq$", let $x \in \text{cone}(X)$. Then, by Caratheodory's theorem there exists a linearly independent set $\widetilde{X} \subseteq X$ such that $x \in \text{cone}(\widetilde{X})$, which concludes the proof.