École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand — November 11, 2016
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Combinatorial Optimization (Fall 2016)

Assignment 6

Deadline: November 18 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Let $M = (X, \mathcal{I})$ be a matroid, prove the following:

1. Every basis B of M (i.e. maximal independent set) has the same cardinality.
2. (Basis exchange property) Given bases B, B' of M, for any $x \in B$ there is a $y \in B'$ such that $B \setminus \{x\} \cup \{y\}$ is a basis.

Solution:

1. Assume there are bases B, B' such that $|B| < |B'|$. But then applying the third axiom of the definition of a matroid we have that there is $b \in B' \setminus B$ such that $B \cup \{b\}$ is independent, a contradiction to the fact that B is maximal independent.
2. Apply part 1 and the third axiom to $B \setminus \{x\}$, B'.

Problem 2 ($\star$)

Let $M = (X, \mathcal{I})$ be a matroid, with $X = \{x_1, \ldots, x_m\}$. Prove that

$$Y = \{x_i \text{ such that } rk(x_1, \ldots, x_i) > rk(x_1, \ldots, x_{i-1})\}$$

is independent (i.e. $Y \in \mathcal{I}$).

Solution:

Let $X_j = \{x_1, \ldots, x_j\}$ and $Y_j = \{x_i : i \leq j, rk(X_i) > rk(X_{i-1})\}$ for $j \leq m$. We shall prove by induction on j that Y_j is independent, for $j \leq m$. The assertion is true for $j = 0$ as $Y_0 = \emptyset$. Suppose $j \geq 1$ and the assertion is true for $j - 1$. There are two cases.
Case 1: $rk(X_j) = rk(X_{j-1})$. In this case, $Y_j = Y_{j-1}$, hence it is independent by induction.
Case 2:. $rk(X_j) > rk(X_{j-1})$. In this case, $Y_j = Y_{j-1} \cup \{x_j\}$. By induction, Y_{j-1} is independent so $|Y_{j-1}| \leq rk(X_{x-1})$, hence Y_{j-1} it is not maximal in X_j, i.e. there is an $x \in X_j \setminus Y_{j-1}$ such that $Y_{j-1} \cup \{x\}$ is independent. But if $x \in X_{j-1}$, then $Y_{j-1} \cup \{x\}$ is still not maximal and we can repeat the argument until we add $x \in X_j \setminus X_{j-1} = \{x_j\}$, which implies that $Y_j = Y_{j-1} \cup \{x_j\}$ is independent.

Alternative solution: We assign weight 1 to each element and we run the greedy algorithm, with the elements of X ordered from x_1 to x_m. Recall that the greedy algorithm starts from the empty set, considers the elements in order and picks the ones that maintain an independent set. Notice that, after step j, the algorithm has picked a maximal independent set for X_j. We will prove that Y is exactly the set picked by the algorithm, hence it is independent (it is actually a basis of M).

Let $Y = \{x_{i_1}, \ldots, x_{i_k}\}$, where $i_1 \leq \cdots \leq i_k$. x_{i_1} is the first element of X to have rank 1, hence it is picked by the greedy algorithm. We now proceed by induction (notice that the argument is analogous to the previous induction): assume the algorithm has picked $Y_{i_{j-1}} = \{x_{i_1}, \ldots, x_{i_{j-1}}\}$, with $j < k$, we show that it must pick x_{i_j}. $\{x_{i_1}, \ldots, x_{i_{j-1}}\}$ is a maximal independent set for $X_{i_{j-1}}$, and x_{i_j} is the first element such that $Y_{i_{j-1}}$ is not maximal in X_{i_j}, hence x_{i_j} belongs to any maximal independent set in X_{i_j}. In particular $Y_{i_j} = Y_{i_{j-1}} \cup \{x_{i_j}\}$ is independent, hence it will be picked by the greedy algorithm.

Problem 3 ($\star$)

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

1. Prove that for any vertex v of P, there is a direction $w \in \mathbb{R}^n$ such that v is the unique optimal solution of the LP $\max\{wx : x \in P\}$.
2. Assume that, for any $w \in \mathbb{R}^n$, the LP $\max\{wx : x \in P\}$ admits as optimal solution an integral vertex. Prove that P is integral (i.e., that all vertices of P have integer coordinates).

Solution:

1. Since v is a vertex of P, there exist an inequality $wx \leq \beta$ such that $wv = \beta$, $wx < \beta$ for any $x \in P, x \neq v$. It suffices to choose such w to get the thesis.
2. Given any vertex v of P, let w be the direction for which v is the unique optimal solution as in part 1. Since $\max\{wx : x \in P\}$ has an integer optimal solution, v must be integer.