École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 28, 2017

Discrete Optimization (Spring 2017)

Assignment 8

Problem 4 can be **submitted** until May 5 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Consider the following polytope:

$$P = \{(x, y, z) : x \geq 0 \quad (1)$$
$$y \geq 0 \quad (2)$$
$$z \geq 0 \quad (3)$$
$$x \leq 1 \quad (4)$$
$$y \leq 1 \quad (5)$$
$$z \leq 1 \; \} \quad (6)$$

and its vertex $v = (0, 0, 0)$. Construct the connected layer family of P with starting point v and compute the diameter of P.

Problem 2

Let $\mathcal{L}$ be a connected layer family and $s \in [n]$ a symbol. Consider the following operation on $\mathcal{L}$, which we call induction on s:

1. remove all vertices from the connected layer family that do not contain s.
2. Remove s from all the remaining vertices.
3. Remove empty layers (and relabel nonempty layers starting from 0).

Prove that the result of the induction operation is again a connected layer family.

Problem 3

The Hirsch Conjecture stated that a polyhedron in d dimensions with n facets ($d < n$) has diameter at most $n - d$. This conjecture was disproved in 2010 by Francisco Santos.
In this exercise, we will construct Hirsch-sharp polyhedra. A polyhedron in d dimensions with n facets ($d < n$) is called *Hirsch-sharp* if it has diameter exactly $n - d$.

(a) Show that for every d and n there is a Hirsch-sharp unbounded polyhedron in d dimensions with n facets.

(b) Show that for every $d < n \leq 2d$ there is a Hirsch-sharp *polytope*.

Problem 4 ($\star$)

Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points $S = \{p \in \mathbb{Z}^n : 0 \leq p_i \leq B \;\forall\; i = 1, \ldots, n\}$, whose points are all colored blue but one, which is red. We have an oracle that, given vectors $l, r \in \mathbb{R}^n$, tells us whether the red point in S is contained in the box

$S \cap \{x \in \mathbb{R}^n : l_i \leq x_i \leq r_i \ \forall\, i = 1, \ldots, n\}$ or not. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

Problem 5

Let $P := \{x \in \mathbb{R}^n : \ Ax = b,\, x \geq 0\}$ be a polyhedron and $\min\{cx : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of A, b and c are integral and bounded in absolute value by given $B \in N$, and furthermore let $L := B^n n^{n/2}$.

(a) Show the following: If x_1, x_2 are vertices of P and $cx_1 \neq cx_2$, then $|cx_1 - cx_2| \geq 1/L^2$.

(b) Let x^* and y^* be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|cx^* - by^*| < 1/L^2$, then each vertex x of P with $cx \leq cx^*$ is an optimal solution of the primal.