École Polytechnique Fédérale de Lausanne
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Combinatorial Optimization (Fall 2016)

Assignment 8

Deadline: December 2 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Prove that G has a perfect matching if and only if for any $W \subseteq V$, the graph $G \setminus W$ obtained by removing W has at most $|W|$ odd components. To prove the "if" direction, use the fact that the system of inequalities describing the matching polytope is Totally Dual Integral.

Solution:
($\rightarrow$) Assume G has a perfect matching M and consider the graph $G \setminus W$ for some $W \subseteq V$. For any odd component in $G \setminus W$, there is at least one vertex in the component that is matched to a vertex in W, hence there cannot be more than W odd components.
($\leftarrow$) We first remark that the hypothesis applied with $W = \emptyset$ implies that $|V|$ is even. Recall that the LP describing the maximum weight matching problem is:

$$\begin{aligned}
\max \sum_{e \in E} w_e x_e & \\
x(\delta(v)) \leq 1 & \qquad \forall\, v \in V \\
x(E(U)) \leq \left\lfloor \frac{|U|}{2} \right\rfloor & \qquad \forall\, U \subseteq V, |U| \text{ odd} \\
x \geq 0 &
\end{aligned}$$

and the corresponding dual is:

$$\begin{aligned}
\min \sum_{v \in V} y_v + \sum_{U} \left\lfloor \frac{|U|}{2} \right\rfloor y_U & \\
y_v + y_w + \sum_{U : e \in E(U)} y_U \geq w_e & \qquad \forall\, e = vw \in E \\
y \geq 0 &
\end{aligned}$$

where we have a variable y_v for each $v \in V$ and a variable y_U for each $U \subseteq V, |U|$ odd . In class we have seen that for any $w \in \mathbb{Z}^E$, the latter LP has an optimal solution with integer coordinates. We set $w_e = 1$ for each $e \in E$. Now the primal corresponds to the problem of finding the matching of maximum cardinality in G, and the optimal value (both of the primal and the dual) is $\frac{|V|}{2}$ if and only if G contains a perfect matching. Let y be the optimal integral solution for the dual corresponding to w, we will make some observations on y and then show that its value is at least $\frac{|V|}{2}$. This means that the value must be exactly $\frac{|V|}{2}$, which concludes the proof.

- We first notice that y has only $0/1$ coordinates. Indeed, $y \geq 0$, and if $y_v \geq 2$ then we could find a feasible solution of strictly smaller value by setting $y_v = 1$ (any constraint in which y_v appears will be still satisfied). The same clearly holds for any y_U.

- We can then define $W = \{v : y_v = 1\}$ and $\mathcal{U} = \{U \subset V : |U| \text{ odd}, \ y_U = 1\}$. For any $U_1, U_2 \in \mathcal{U}$, we cannot have that $U_1 \subset U_2$: indeed, setting $y_{U_1} = 0$ would again yield a feasible solution of strictly smaller value.

- More generally, we now show that can assume that for any $U_1, U_2 \in \mathcal{U}$, $U_1 \cap U_2 = \emptyset$. First assume that $|U_1 \cap U_2|$ is odd. Modify y by setting $y_{U_1} = y_{U_2} = 0$ and $y_{U_1 \cup U_2} = 1$ ($|U_1 \cup U_2|$ is odd). The solution is still feasible, and the objective value did not increase, since:

$$\left\lfloor \frac{|U_1|}{2} \right\rfloor + \left\lfloor \frac{|U_2|}{2} \right\rfloor = \frac{|U_1| + |U_2| - 2}{2} \geq \frac{|U_1| + |U_2| - |U_1 \cap U_2| - 1}{2} = \left\lfloor \frac{|U_1 \cup U_2|}{2} \right\rfloor.$$

  Now, if $|U_1 \cap U_2| \geq 2$ and is even, we can repeat a similar argument: in the case that $U_1 \cup U_2 \neq V$, set $y_{U_1 \cup U_2 \cup \{v\}} = 1$, where v is any vertex in $V \setminus U_1 \cup U_2$; otherwise, choose any $v \in V$ and set $y_{V \setminus \{v\}} = 1$ and $y_v = 1$.

- For any $U \in \mathcal{U}$, we can assume that $U \cap W = \emptyset$. Indeed, if this is not the case, there are two cases: if $|U \cap W|$ is even, then $U \setminus W$ has odd cardinality and we can set $y_U = 0, y_{U \setminus W} = 1$, and get a feasible solution of strictly smaller value; if $|U \cap W|$ is odd, then choose a vertex $v \in U \setminus W$, set $y_v = 1$ and $y_{U \setminus (W \cup \{v\})} = 1$.

- Now, since no edge can lie between two sets of $\mathcal{U}$, we have that the sets in $\mathcal{U}$ are components (or unions of components) of $G \setminus W$: indeed, if $U_1, U_2 \in \mathcal{U}$ contain vertices of the same component, either they intersect, which we have excluded, or they leave at least one edge "uncovered", but then this edge must be covered by a third set in $\mathcal{U}$ and we repeat the argument. We now show that each $U \in \mathcal{U}$ contains exactly one component of $G \setminus W$ with odd cardinality. First notice that the number of odd components in U must be odd. If there are $C_1, C_2, C_3 \subset U$ with odd cardinality, we can see that similarly as before setting $y_{C_1} = y_{C_2} = y_{U \setminus \{C_1 \cup C_2\}} = 1$ and $y_U = 0$ gives us a solution of strictly smaller value than before, which is a contradiction:

$$\left\lfloor \frac{|C_1|}{2} \right\rfloor + \left\lfloor \frac{|C_2|}{2} \right\rfloor + \left\lfloor \frac{|U \setminus \{C_1 \cup C_2\}|}{2} \right\rfloor = \frac{|C_1| - 1 + |C_2| - 1 + |U \setminus \{C_1 \cup C_2\}| - 1}{2} < \left\lfloor \frac{|U|}{2} \right\rfloor.$$

Finally, without loss of generality, we have $\mathcal{U} = \{U_1, \ldots, U_k\}$ with each U_i containing exactly one odd component for $i < k$, hence by hypothesis $k \leq |W|$. But then we have:

$$\sum_{v \in V} y_v + \sum_{U \in \mathcal{U}} \left\lfloor \frac{|U|}{2} \right\rfloor y_U = |W| + \sum_{i=1}^{k} \left\lfloor \frac{|U_i|}{2} \right\rfloor =$$

$$= |W| + \sum_{i=1}^{k} \frac{|U_i| - 1}{2} = |W| + \frac{|V| - |W| - k}{2} \geq \frac{|V|}{2},$$

which concludes the proof.

Problem 2

We saw that the description of the matching polytope of a graph $G(V, E)$ contains an inequality for each $U \subseteq V$ such that $|U|$ is odd. If $|V| = n$, how many such inequalities are there? Is there any of those inequalities that is clearly redundant? (An inequality is redundant if it can be removed from the description without changing the polytope).

Solution:

We just need to count the number of odd subsets of V, which is equal to: $n + \binom{n}{3} + \binom{n}{5} + \cdots = 2^n - 1 - \binom{n}{2} - \binom{n}{4} - \ldots$. We now use the Netwon Bynomial Theorem to get: $\sum_{i=0}^{n} \binom{n}{i} = (1+1)^n = 2^n$, and $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = (1-1)^n = 0$, which imply that $n + \binom{n}{3} + \binom{n}{5} + \cdots = 1 + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$.

The n inequalities stemming from $U = \{v\}$ for some $v \in V$ are clearly redundant since they are of the form $0 \leq 0$.

Problem 3

Prove the following lemma used in class: let $G(V, E)$ be a connected graph and $w : E \to \mathbb{N}_{\geq 1}$, at least one of the following must hold:

a) There is a vertex $v \in V$ such that $\delta(v) \cap M \neq \emptyset$ for any $M \in \mathcal{M}(w)$.

b) $z(w') = z(w) - \left\lfloor \frac{|V|}{2} \right\rfloor$, and $|V|$ is odd.

Where: $w' = w - \vec{1}$ is the vector w with each entry decreased by 1; $\mathcal{M}(w)$ is the set of maximum weight matchings with respect to w (similarly for w'); $z(w) = w(M)$ for any $M \in \mathcal{M}(w)$ (similarly for w').

Solution:
In the lecture, we proved the following lemma:
Lemma 1 Let $G(V, E)$ be a connected graph and $w : E \to \mathbb{R}_{>0}$, at least one of the following must hold:

1. There is a vertex $v \in V$ such that $\delta(v) \cap M \neq \emptyset$ for any $M \in \mathcal{M}(w)$
2. For any $M \in \mathcal{M}(w)$, $|M| = \left\lfloor \frac{|V|}{2} \right\rfloor$ and $|V|$ is odd.

We will mimic the proof of this lemma, and also use the result itself. Suppose that both a), b) do not hold. We first remark that we might have edges e such that $w(e) = 1$, hence $w'(e) = 0$; to avoid confusion we restrict $\mathcal{M}(w')$ to matchings of maximum cardinality, i.e. we allow matchings to include edges of weight 0. Now, since a) and the case 1) are the same, by Lemma 1 we have that 2) must hold, in particular $|V|$ is odd. Hence for any $M \in \mathcal{M}(w')$, $w'(M) > z(w) - \left\lfloor \frac{|V|}{2} \right\rfloor$, but this implies that $|M| < \left\lfloor \frac{|V|}{2} \right\rfloor$ hence M has at least two exposed nodes. We choose M, and two exposed nodes u, v such that they achieve the minimal distance. u, v cannot be adjacent, otherwise adding the edge uv to M would increase its weight. Hence the shortest path between u and v has at least a third vertex, which we denote by t. By minimality of the distance between u and v, t must be matched by M. However since a) doesn't hold there exist $M' \in \mathcal{M}(w)$ such that t is exposed in M'. Moreover, since 2) holds, $|M'| = \left\lfloor \frac{|V|}{2} \right\rfloor$, hence t is the only exposed node for M'. Now, in $M \Delta M'$, t is the starting point of an alternating path P that must end in an exposed node: this node must be exposed for M, hence P is an even path (it starts with an edge of M and ends with an edge of M'). Now we consider $\bar{M} = M \setminus (P \cap M) \cup (P \setminus M)$, the matching obtained from M by flipping the edges of P, and similarly $\bar{M}' = M' \setminus (P \cap M') \cup (P \setminus M') = M' \setminus (P \setminus M) \cup (P \cap M)$. We claim that $\bar{M} \in \mathcal{M}(w')$: if this is true, we are done because in $\bar{M}$ t is exposed, as well as at least one of u, v, contradicting the minimality of the distance between u, v (similarly as in the proof of Lemma 1).
We have $w'(\bar{M}) = w'(M) - w'(M \cap P) + w'(P \setminus M)$, hence we need to show that $w'(M \cap P) = w'(P \setminus M)$. We have that $w'(M \cap P) \geq w'(P \setminus M)$, since $M \in \mathcal{M}(w')$. Assume strict inequality holds. But then

$$w(\bar{M}') = w(M') - w'(P \setminus M) + \frac{|P|}{2} + w'(P \cap M) - \frac{|P|}{2} > w(M')$$

a contradiction to the fact that $M' \in \mathcal{M}(w)$. Hence the claim is proved.