Prof. F. Eisenbrand EPFL - DISOPT

Combinatorial Optimization

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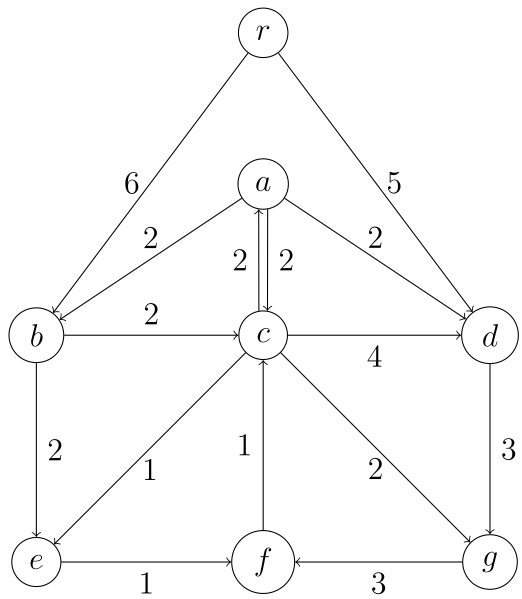
Sheet 6 December 1, 2011

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercises marked with a star at the beginning of the exercise session on December 13.

Exercise 1
Trace the steps of algorithm from the lecture to compute a minimum weight arborescence rooted at r in the following example.



Prove the optimality of your solution!

Exercise 2

Let $G = (A \cup B, E)$ be a bipartite graph. We define two partition matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ with $\mathcal{I}_1 = \{I \subseteq E \colon |I \cap \delta(a)| \leq 1 \quad \text{for all } a \in A\}$ and $\mathcal{I}_2 = \{I \subseteq E \colon |I \cap \delta(b)| \leq 1 \quad \text{for all } b \in B\}$. What is a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ in terms of graph theory? Can you maximize a weight function $w : E \to \mathbb{R}$ over the intersection?

Remark: This is a special case of the optimization over the intersection of two matroids. It can be shown that all such matroid intersection problems can be solved efficiently.

Exercise 3 ($\star$)

Recall that a digraph $D = (V, A)$ is called a branching in D if the underlying undirected graph is a forest and each vertex $v \in V$ has at most one incoming arc.

(i) Let $D = (V, A)$ be a digraph and let $\mathcal{B}$ be a set of all branchings in D (i.e., subsets $B \subseteq A$ such that (V, B) is a branching). Show that $(A, \mathcal{B})$ is an intersection of two matroids.

(ii) Let $r \in V$. Show how to model the arborescences rooted at r using the intersection of two matroids.

Exercise 4

Describe a linear time algorithm which for any instance of the Satisfiability problem finds a truth assignment that satisfies at least half of the clauses.

Exercise 5 ($\star$)

Show that deciding if a polyhedron contains an integer point is NP-complete. To do so, consider the following problem:

INTEGER LINEAR INEQUALITIES
Given: a matrix $A \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$
Task: Is there a vector $x \in \mathbb{Z}^n$ such that $Ax \leq b$?

Show that

(i) Integer Linear Inequalities is in NP.

(ii) Integer Linear Inequalities is NP-complete.

Exercise 6

Consider the following problem:

DOMINATING SET
Given: an undirected graph $G = (V, E)$ and a number $k \in \mathbb{N}$
Task: Is there a set $X \subseteq V$ with $\lvert X\rvert \leq k$ and for every $v \in V \setminus X$, we have $\{x, v\} \in E$?

Show that

(i) Dominating Set is in NP.

(ii) Dominating Set is NP-complete.

Hint: Vertex Cover