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Exercises

Approximation Algorithms

Spring 2010

Sheet 6

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 1
Let $I = (a_1, \ldots, a_n)$ with $a_i \in [0,1]$ be a BIN PACKING instance. Consider the following algorithm

(1) Apply linear grouping with parameter k and call the emerging instance $I' = (a'_1, \ldots, a'_k)$ (Item a'_i appears $b_i \in \mathbb{N}_0$ times)

(2) Compute a near-optimal basic solution x of the Gilmore Gomory LP-relaxation for I'

(3) Buy $\lceil x_p \rceil$ times pattern $p \in \mathscr{P}$

Perform the following tasks:

i) Show that for a suitable choice of k, the above algorithm produces a solution that needs at most $OPT + O(\sqrt{n})$ bins.

ii) An *asymptotic FPTAS* for BIN PACKING is an algorithm that for any given $\varepsilon > 0$ finds a solution with at most $(1+\varepsilon)OPT + p(1/\varepsilon)$ bins where the running time must be polynomial in n and $1/\varepsilon$. Furthermore $p : \mathbb{R}_+ \to \mathbb{R}_+$ must be a polynomial. Show that if you run the above algorithm on the large items and distribute the small items afterwards (as usual), one obtains such an asymptotic FPTAS.
Hints: You will need a suitable threshold, to determine what a *small* item is.

Solution:

i) From the lecture we know that $OPT(I') \leq OPT(I) + \lceil n/k \rceil$. Say we obtain a solution x with $\mathbf{1}^T x \leq OPT_f(I') + 1 \leq OPT(I) + 1$. Furthermore $|\{p \mid x_p > 0\}| \leq k$. Hence we buy

$$OPT(I') + 1 + k \leq OPT(I) + 1 + k + \lceil \frac{n}{k} \rceil \overset{k:=\lceil\sqrt{n}\rceil}{\leq} OPT(I) + 4\sqrt{n}$$

patterns.

ii) We first discard items that have size $a_i < \frac{1}{n^{1/3}}$ and distribute them afterwards. In case that a new bin needs to be opened and the solution consists of m bins, we have

$$size \geq (m-1)\cdot(1-\frac{1}{n^{1/3}})$$

Hence

$$m \leq (1+2\frac{1}{n^{1/3}})OPT+1$$

If $\varepsilon \geq 2\frac{1}{n^{1/3}}$ there is nothing to show. Hence suppose that $\varepsilon < 2\frac{1}{n^{1/3}}$. Then

$$2\frac{1}{n^{1/3}}OPT+1 \leq 2n^{2/3}+1 \leq 256\cdot(2/n^{1/3})^3$$

Next, let $\varepsilon > 0$ be given and choose $p(1/\varepsilon) = 256/\varepsilon^3$. Suppose that $a_i \geq \frac{1}{n^{1/3}}$. Hence $OPT \geq n\cdot\frac{1}{n^{1/3}} = n^{2/3}$. If $\varepsilon \geq 4\frac{1}{n^{1/6}}$ then

$$\varepsilon OPT \geq 4\cdot\frac{1}{n^{1/6}}\cdot n^{2/3} = 4\sqrt{n}$$

If $\varepsilon < 4\frac{1}{n^{1/6}}$ then

$$p(1/\varepsilon) \geq 256\cdot(n^{1/6}/4)^3 = 4\sqrt{n}$$

Hence

$$OPT + 4\sqrt{n} \leq OPT + \varepsilon OPT + p(1/\varepsilon).$$

Exercise 2

Again consider the BIN COVERING problem on instance $I = (a_1,\ldots,a_n)$ ($a_i \in [0,1]$) with the restriction that $a_i \geq \delta$ for a universal constant $\delta > 0$. Adapt the algorithm from the previous exercise to obtain a solution covering $OPT - O(\sqrt{n})$ bins in polynomial time.
Hints: How would an adapted Gilmore-Gomory LP-relaxation look like? Show that under the assumption $a_i \geq \delta$ you can solve it optimally.

Solution:

First we apply the linear grouping to I to obtain a rounded down instance I' with item sizes $a'_1,\ldots,a'_k$ and multiplicities $b_1,\ldots,b_k$ and $OPT(I') \geq OPT(I) - n/k$. We define covering patterns $\mathscr{P} = \{p \in \mathbb{Z}^n_+ \mid a^T p \geq 1, \sum_{i=1}^k p_i \leq 1/\delta\}$ and define

$$\begin{aligned} \max \mathbf{1}^T x & \qquad (P(\mathscr{P})) \\ \sum_{p\in\mathscr{P}} x_p p & \leq b \\ x & \geq \mathbf{0} \end{aligned}$$

Note that we do not need to consider patterns with more than $1/\delta$ many (suppose $\frac{1}{\delta}$) items (otherwise we can remove one and still cover the bin). Hence $|\mathscr{P}| \leq O(k^{1/\delta}) = O(n^{1/\delta})$ which is polynomial (since we consider $\delta > 0$ to be fixed). Consider the following algorithm

(1) Apply linear grouping with parameter k to all items and call the emerging instance $I' = (a'_1,\ldots,a'_k)$ (Item a'_i appears $b_i \in \mathbb{N}_0$ times)

(2) Compute an optimal basic solution x to $P(\mathscr{P})$ (for I')

(3) Buy $\lfloor x_p \rfloor$ times pattern $p \in \mathscr{P}$

The number of bins, that we cover with our solution is at least

$$P(\mathscr{P}) - |\{p \mid x_p > 0\}| \geq OPT(I') - k \geq OPT(I) - \lceil n/k \rceil - k \stackrel{k:=\lceil\sqrt{n}\rceil}{=} OPT(I) - 4\sqrt{n}$$
