

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 4

Exercise 4.1
Let $Q \in \mathbb{R}^{n\times n}$ be a positive-definite matrix (i.e. $\forall x \neq \mathbf{0} : x^T Qx > 0$) and $c \in \mathbb{R}^n$ be a vector. Then $\min\{x^T Qx + c^T x \mid x \in \mathbb{R}^n\}$ is bounded, i.e. there exists an M such that $x^T Qx + c^T x \geq -M$ for all $x \in \mathbb{R}^n$.

Exercise 4.2
Let $P \subseteq \mathbb{R}^n$ be a polytope, Q be a symmetric, positive semidefinite matrix, $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := x^T Qx$, $x^* = \text{argmin}\{f(x) \mid x \in P\}$. Define

$$C := \max\left\{(x-y)^T Q(x'-y') \mid x,y,x',y' \in P \cup \{\mathbf{0}\}\right\}$$ [1]

and $h(x) = \frac{f(x)-f(x^*)}{4C}$. Prove that $h(x^{(0)}) \leq 1/4$, where $x^{(0)}$ is an arbitrary point in P.

Exercise 4.3
Let $P \subseteq [-M,M]^n$ be a polytope and $Q \in [-M,M]^{n\times n}$ be a symmetric, positive semidefinite matrix. Give a bound[2] (depending on n,M,ε) on the number of iterations k, that the Frank-Wolfe algorithm needs, to reach a solution $x^{(k)}$ such that $f(x^{(k)}) - f(x^*) \leq \varepsilon$ (with $f(x) := x^T Qx$).

Exercise 4.4
Let $P \subseteq \mathbb{R}^n$ be a polytope, Q be a symmetric, positive semidefinite matrix, $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) := x^T Qx$, $w(y) := \min_{v\in P}\{\nabla f(y)^T (v-y) + f(y)\}$,

$$C := \max\left\{(x-y)^T Q(x'-y') \mid x,y,x',y' \in P \cup \{\mathbf{0}\}\right\}$$

Let $\lambda^* = \frac{f(x^{(k)})-w(x^{(k)})}{2C}$ minimizing $g(\lambda) = f(x^{(k)} + \lambda(y^{(k)} - x^{(k)}))$. Prove that $\lambda^* \in [0,1]$.

Exercise 4.5
Prove that

$$\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} \leq e^{-\frac{1}{2(n+1)}}$$

for all $n \geq 2$.

[1] We define C here differently then in the lecture to avoid some technical difficulties
[2] the bound does not need to be the best possible one