The matching polytope

We now come to a deeper theorem concerning the convex hull of matchings. We mentioned several times in the course that the maximum weight matching problem can be solved in polynomial time. We are now going to show a theorem of Edmonds [1] which provides a complete description of the matching polytope and present the proof by Lovász [3].

Before we proceed let us inspect the symmetric difference $M_1 \Delta M_2$ of two matrchings of a graph G. If a vertex is adjacent to two edges of $M_1 \cup M_2$, then one of the two edges belongs to M_1 and one belongs to M_2. Also, a vertex can never be adjacent to three edges in $M_1 \cup M_2$. Edges which are both in M_1 and M_2 do not appear in the symmetric difference. We therefore have the following lemma.

Lemma 16. *The symmetric difference $M_1 \Delta M_2$ of two matchings decomposes into node-disjoint paths and cycles, where the edges on these paths and cycles alternate between M_1 and M_2.*

The *Matching polytope* $P(G)$ of an undirected graph $G = (V, E)$ is the convex hull of incidence vectors χ^M of matchings M of G.

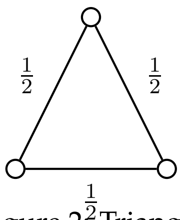


Figure 2: Triangle

The incidence vectors of matchings are exactly the $0/1$-vectors that satisfy the following system of equations.

$$\begin{array}{rcll} \sum_{e \in \delta(v)} x(e) & \leqslant & 1 & \forall v \in V \\ x(e) & \geqslant & 0 & \forall e \in E. \end{array} \tag{10}$$

However the triangle (Figure 2) shows that the corresponding polytope is not integral. The objective function $\max 1^T x$ has value 1.5. However, one can show that a maximum weight matching of an undirected graph can be computed in polynomial time which is a result of Edmonds [2].

The following (Figure 3) is an illustration of an Edmonds inequality. Suppose that U is an odd subset of the nodes V of G and let M be a matching of G. The number of edges of M with both endpoints in U is bounded from above by $\lfloor |U|/2 \rfloor$.

Thus the following inequality is valid for the integer points of the polyhedron defined by (10).

$$\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor, \qquad \text{for each } U \subseteq V, \quad |U| \equiv 1 \pmod 2. \tag{11}$$

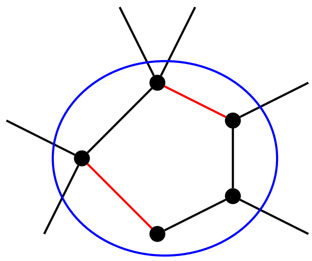


Figure 3: Edmonds inequality.

The goal of this lecture is a proof of the following theorem.

Theorem 17 (Edmonds 65)**.** *The matching polytope is described by the following inequalities:*

i) $x(e) \geqslant 0$ *for each* $e \in E$*,*

ii) $\sum_{e \in \delta(v)} x(e) \leqslant 1$ *for each* $v \in V$*,*

iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ *for each* $U \subseteq V$

Lemma 18. *Let* $G = (V, E)$ *be connected and let* $w : E \longrightarrow \mathbb{R}_{>0}$ *be a weight-function. Denote the set of maximum weight matchings of G w.r.t.* w *by* $\mathscr{M}(w)$*. One has*

i) $\exists\, v \in V$ *such that* $\delta(v) \cap M \neq \emptyset$ *for each* $M \in \mathscr{M}(w)$

ii) $|M| = \lfloor |V|/2 \rfloor$ *for each* $M \in \mathscr{M}(w)$ *and* $|V|$ *is odd*

Proof. Suppose $i)$ and $ii)$ do not hold Then there exists $M \in \mathscr{M}(w)$ leaving two exposed nodes u and v. Choose M such that the minimum distance between two exposed nodes u, v is minimized.

Now let t be on shortest path from u to v. The vertex t cannot be exposed.

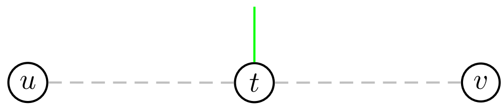
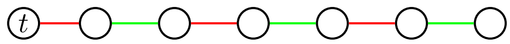


Figure 4: Shortest path between u and v.

Let $M' \in \mathscr{M}(w)$ leave t exposed. Both u and v are covered by M' because the distance to u or v from t is smaller than the distance of u to v.

Consider the symmetric difference $M \triangle M'$ which decomposes into node disjoint paths and cycles. The nodes u, v and t have degree one in $M \triangle M'$. Let P be a path with endpoint t in $M \triangle M'$



If we swap colors on P we obtain matchings $\widetilde{M}$ and $\widetilde{M'}$ with $w(M) + w(M') = w(\widetilde{M}) + w(\widetilde{M'})$ and thus $\widetilde{M} \in \mathscr{M}(w)$.

The node t is exposed in $\widetilde{M}$ and u or v is exposed in $\widetilde{M}$. This is a contradiction to u and v being shortest distance exposed vertices

□

Proof of Theorem 17. Let $w^T x \leqslant \beta$ be a *facet* of $P(G)$, we need to show that this facet it is of the form

i) $x(e) \geqslant 0$ for some $e \in E$

ii) $\sum_{e \in \delta(v)} x(e) \leqslant 1$ for some $v \in V$

iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ for some $U \in P_{odd}$

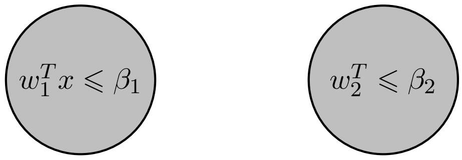
To do so, we use the following method: One of the inequalities i), ii), iii) is satisfied with equality by each χ^M, $M \in \mathscr{M}(w)$. This establishes the claim since the matching polytope is full-dimensional and a facet is a maximal face.

If $w(e) < 0$ for some $e \in E$, then each $M \in \mathscr{M}(w)$ satisfies $e \notin M$ and thus satisfies $x(e) \geqslant 0$ with equality.

Thus we can assume that $w \geqslant 0$.

Let $G^* = (V^*, E^*)$ be the graph induced by edges e with $w(e) > 0$. Each $M \in \mathscr{M}(w)$ contains maximum weight matching $M^* = M \cap E^*$ of G^* w.r.t. w^*.

If G^* is not *connected* , suppose that $V^* = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$ and there is no edge connecting V_1 and V_2, then $w^T x \leqslant \beta$ can be written as the sum of $w_1^T x \leqslant \beta_1$ and $w_2^T x \leqslant \beta_2$, where β_i is the maximum weight of a matching in V_i w.r.t. w_i, $i = 1, 2$. This would also contradict the fact that $w^T x \leqslant \beta$ is a facet, since it would follow from the previous inequalities and thus would be a redundant inequality.



Now we can use Lemma 18for G^*.

i) $\exists v$ such that $\delta(v) \cap M \neq \emptyset$ for each $M \in \mathscr{M}(w)$. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e \in \delta(v)} x(e) \leqslant 1 \quad \text{with equality}$$

ii) $|M \cap E^*| = \lfloor |V^*|/2 \rfloor$ for each $M \in \mathscr{M}(w)$ and $|V^*|$ is odd. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e \in E(V^*)} x(e) \leqslant \lfloor |V^*|/2 \rfloor \quad \text{with equality}$$

□

Bibliography

[1] J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards*, 69:125–130, 1965.

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