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Discussions from: September 18, 2015

Combinatorial Optimization

Fall 2015

Assignment Sheet 1

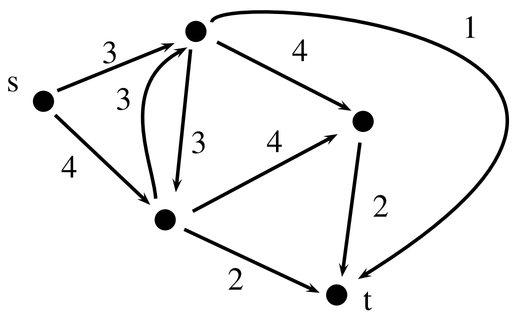
Exercises marked with a ⋆ can be handed in for bonus points. Due date is September 25.

Recall the max-flow algorithm seen in class.

Given: a digraph $G(V,A)$ with capacities $u : A \to Q_{\geq 0}$, and a pair of distinct vertices $s, t \in V$.

- Set $f = 0$ and construct the auxiliary graph D_f.
- While there is an oriented path between s and t in D_f:
  - Augment f to f' (*augmenting step*).
  - Set $f = f'$.
  - Construct the auxiliary graph D_f.
- Output f.

Exercise 1
Using the algorithm above, compute a max $s-t$ flow in the directed capacitated graph below.



Exercise 2
Prove that, if the capacities are integral, then there exists a maximum $s-t$ flow that is an integral vector.

Exercise 3
In this exercise we will show that the max-flow algorithm seen in class has running time $O(|V||A|^2)$, if each time in the augmenting step, we augment along a shortest path (i.e. a path with a minimum number of edges) between s and t.

a) [$\star$] In a digraph $D(V, A)$, let $\mu(D)$ be the length of a shortest path between s and t and $\alpha(D)$ the set of edges that are in at least a shortest path between s and t. Prove that $\mu(D') = \mu(D)$ and $\alpha(D') = \alpha(D)$, where $D' = (V, A \cup \alpha(D)^{-1})$.

b) Using the previous part, show that the number of augmenting steps of the max-flow algorithm is bounded by $|A||V|$.
(Hint: consider the residual graph $D_{f'}$, where f' is the flow after the augmentation. How does $\mu(D_{f'})$ and $\alpha(D_{f'})$ change with respect to $\mu(D_f)$, $\alpha(D_f)$?)

c) Assuming that a shortest path in D_f can be found in time $O(|A|)$, conclude that the running time of the max flow algorithm is $O(|V||A|^2)$.

Exercise 4

Provide a family of instances showing that a wrong strategy in choosing an $s-t$ path in D_f may lead to a non-polynomial (in $|V|$) algorithm for max-flow.