
The problem can be submitted until March 15, 12 :00 noon, either at the exercise session or into the box in front of MA C1 563.

Student(s) [1] :

Question 1 : *The question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5	*Reserved for the corrector*

Consider a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ with $A \in \mathbb{R}^{m\times n}$ full column rank and a feasible basis $B \subseteq \{1, \dots, m\}$. (Recall that a basis is feasible if $x_B^* = A_B^{-1} b_B$ is a feasible solution, in fact a vertex.) The aim is to show that, if B is not an optimal basis, x_B^* is not an optimal solution under a certain condition.

(i) Show that there exists a unique $\lambda \in \mathbb{R}^m$ such that

$$\lambda^T A = c^T \text{ and } \lambda_j = 0 \text{ for each } j \notin B.$$

(ii) Let $i \in B$. Show that there exists a unique $d_i \in \mathbb{R}^n$, $d_i \neq 0$ such that

$$a_j^T d_i = \begin{cases} 0 & \text{for } j \in B \setminus \{i\} \\ -1 & \text{if } j = i. \end{cases}$$

Show that $\lambda_i < 0$ implies $c^T d_i > 0$.

(iii) Conclude that, if the inequalities that are tight at x_B^* are those indexed by B only, then x_B^* is not optimal.

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1. You are allowed to submit your solutions in groups of at most three students.

Sol.:

(i) Consider the matrix A_B given by the basis B. Since B is feasible, it is of full rank and we can write $\lambda_B = c^T A_B^{-1}$. Note that λ_B is in $\mathbb{R}^n$ and it is the unique vector with this property. To get λ we complete λ_B to a vector in $\mathbb{R}^m$ by adding zeros for all elements not in B :

$$\lambda_j = \begin{cases} (\lambda_B)_j & \text{for } j \in B \\ 0 & \text{if } j \notin B. \end{cases}$$

The uniqueness of λ is given by the fact that λ_B is unique.

(ii) Fix $i \in B$. The vector d_i is a solution to the linear program $A_B x = -e_i$. Since A_B is a basis, it is invertible and we get $d_i = -A_B^{-1} e_i$, which is the i-th column of $-A_B^{-1}$. This gives the existence and uniqueness of d_i.
For the second part, let i be a coefficient with $\lambda_i < 0$. Then :

$$c^T d_i = (\lambda_B^T A_B)(-A_B^{-1} e_i) = -\lambda_B^T e_i = -\lambda_i > 0$$

(iii) Suppose that the only tight inequalities at x_B^* are those indexed by B. Since B is not an optimal basis, $\lambda \ngeq 0$ which implies that there exists an index i with $\lambda_i < 0$.
Consider d_i as described in (ii). Note that non of the inequalities $a_j^T x \leq b_j$ are tight at x_B^* for all $j \notin B$. So for every vector $v \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that $a_j^T (x_B^* + \varepsilon v) \leq b_j$ for all $j \notin B$. This is true in particular for the vector d_i. By the way d_i was chosen, for all $i \neq j \in B$ we get $a_j^T (x_B^* + \varepsilon d_i) = a_j^T x_B^* + \varepsilon a_j^T d_i = b_j + 0 \leq b_j$ and $a_i^T (x_B^* + \varepsilon d_i) = a_i^T x_B^* + \varepsilon a_i^T d_i = b_i - \varepsilon \leq b_i$. Thus the point $(x_B^* + \varepsilon d_i)$ is a feasible point of our polytope $Ax \leq b$.
It remains to show that the objective improves at $x_B^* + \varepsilon d_i$. This is easy to calculate since $c^T d_i > 0$ and so we get $c^T (x_B^* + \varepsilon d_i) = c^T x_B^* + \varepsilon c^T d_i > c^T x_B^*$