The problem can be submitted until April 5, 12 :00 noon, either at the exercise session or into the box in front of MA C1 563.

Student(s) [1] :

Question 1 : *The question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

Given a graph $G = (V, E)$ with a weight function $w : V \to \mathbb{R}$ on its vertices.
Consider the following linear program and its dual.

Primal				**Dual**			
$\min \sum_{v \in V} w(v) x_v$				$\max \sum_{e \in E} y_e$			
$x_u + x_v$	$\geq$	1	$\forall \{u, v\} \in E$	$\sum_{e \in E, e \ni v} y_e$	$\leq$	$w(v)$	$\forall v \in V$
x_v	$\geq$	0	$\forall v \in V$	y_e	$\geq$	0	$\forall e \in E$

Define $C_y \subseteq V$ to be the set of vertices for which the corresponding dual constraints are tight in y, i.e. $C_y = \{v \in V : \sum_{e \in E, e \ni v} y_e = w(v)\}$.
We apply the following algorithm for the dual linear program :

- Initialize the dual solution y to be $y_e = 0$ for every $e \in E$
- While there exists $\{u, v\} \in E$ such that $C_y \cap \{u, v\} = \emptyset$:
  Increase $y_{u,v}$ until one of the dual constraints (corresponding to u or v) becomes tight.
- Return y

Let y be the solution given by the algorithm and C_y its corresponding set of vertices. Show that for an optimal solution x^* of the primal linear program :

$$\sum_{v \in C_y} w(v) \leq 2 \sum_{v \in V} w(v) x_v^*$$

1. You are allowed to submit your solutions in groups of at most three students.

Sol.:
First we note that the algorithm outputs a feasible solution of the dual linear program. In fact, the initial solution is clearly feasible and during one step of the algorithm, feasibility is maintained. This is because we only augment the $y_{u,v}$ if it did not appear in a tight inequality and we augment until the first inequality becomes tight, which implies that all inequalities remain satisfied.
Now, let x^* be an optimal solution of the primal linear program, y the solution of the dual linear program given by the algorithm and C_y the set of vertices corresponding to y.
We use that for each $v \in C_y$, by definition $w(v) = \sum_{e\in E, v\in e} y_e$:

$$\sum_{v\in C} w(v) = \sum_{v\in C} \sum_{e\in E, v\in e} y_e = \sum_{e\in E} \sum_{v\in C, v\in e} y_e$$

For a fixed $e \in E$ the sum $\sum_{v\in C, v\in e} y_e$ contains at most 2 terms since e has two vertices. Thus we can bound the sum by $2y_e$ to get :

$$\sum_{v\in C} w(v) \le \sum_{e\in E} 2y_e = 2\sum_{e\in E} y_e$$

Note that $\sum_{e\in E} y_e$ is the objective function of the dual linear program. By weak duality we get that for each feasible solution y' of the dual linear program and x' of the primal linear program $\sum_{e\in E} y'_e \le \sum_{v\in V} w(v)x'_v$. In particular, this is true for the two feasible solutions x^* and y of the primal respectively the dual linear program. We get :

$$\sum_{v\in C} w(v) \le 2\sum_{e\in E} y_e \le 2\sum_{v\in V} w(v)x^*_v$$

Which is exactly what we wanted to prove.