Graph theory - solutions to problem set 1

1. Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ we define the *degree sequence* of G to be the list $d(v_1), \ldots, d(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:

   (a) $3, 3, 2, 2, 2, 1$

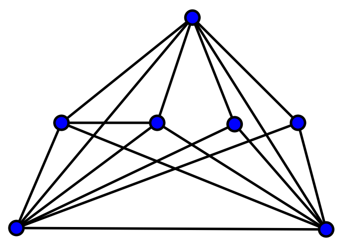
   (b) $6, 6, 6, 4, 4, 3, 3$

   (c) $6, 6, 6, 4, 4, 2, 2$

   Solution:

   (a) There is no such graph; since by problem 5, the number of odd-degree vertices in a graph is always even.

   (b) Consider the following graph:

   (c) No, since otherwise we have 3 vertices of degree 6 which are adjacent to all other vertices of the graph; so each vertex in the graph must be of degree at least 3.

2. Construct two graphs that have the same degree sequence but are not isomorphic.

   Solution: Let G_1 be of a cycle on 6 vertices, and let G_2 be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

3. A graph is *k-regular* if every vertex has degree k. How do 1-regular graphs look like? And 2-regular graphs?

   Solution: A 1-regular graph is just a disjoint union of edges (soon to be called a matching). A 2-regular graph is a disjoint union of cycles.

4. How many (labelled) graphs exist on a given set of n vertices? How many of them contain exactly m edges?

   Solution: Since there are $\binom{n}{2}$ possible edges on n vertices, and a graph may or may not have each of these edges, we get that there are $2^{\binom{n}{2}}$ possible graphs on n vertices. For the second problem, out of the $\binom{n}{2}$ possible edges, we want to choose m ones. So there are $\binom{\binom{n}{2}}{m}$ possible graphs on n vertices and with m edges.

5. Prove that the number of odd-degree vertices in a graph is always even.

   Solution: Let $G = (V, E)$ be an arbitrary graph. In the lecture we have proved that $\sum_{v \in V} d(v) = 2|E|$. Let $V_1 \subseteq V$ be the set of vertices of G which have odd degree and $V_2 = V \backslash V_1$ be the set of vertices of G which have even degree. We have that

$$\sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2|E|.$$

Since all the vertices in V_2 have even degree, and $2|E|$ is even, we obtain that $\sum_{v\in V_1} d(v)$ is even. But since V_1 is the set of vertices of odd degree, we obtain that the cardinality of V_1 is even (that is, there are an even number of vertices of odd degree), which completes the proof.

6. Let G be a graph with minimum degree $\delta > 1$. Prove that G contains a cycle of length at least $\delta + 1$.

   Solution: First, let's recall how we proceeded in the lecture to find a path of length at least δ:

   Let $v_1 \cdots v_k$ be a maximal path in G, i.e., a path that cannot be extended. Then any neighbor of v_1 must be on the path, since otherwise we could extend it. Since v_1 has at least $\delta(G)$ neighbors, the set $\{v_2, \ldots, v_k\}$ must contain at least $\delta(G)$ elements. Hence $k \geq \delta(G) + 1$, so the path has length at least $\delta(G)$.

   Now in order to find a cycle of length at least $\delta + 1$, we continue the proof above. The neighbor of v_1 that is furthest along the path must be v_i with $i \geq \delta(G) + 1$. Then $v_1 \cdots v_i v_1$ is a cycle of length at least $\delta(G) + 1$.

7. Show that every graph on at least two vertices contains two vertices of equal degree.

   Solution: Suppose that the n vertices all have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n - 1$, the set of degrees must be

$$\{0, 1, 2, \ldots, n-2, n-1\}.$$

   But that's not possible, because the vertex with degree $n - 1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

8. Prove that at a meeting of at least 6 people, there are always 3 that mutually know each other, or 3 that mutually do not know each other.

   Hint: start by proving the following statement. If G is a graph on at least 6 vertices, then either G or its complement has a vertex of degree at least 3.

   The complement of a graph $G = (V, E)$, denoted G^C, is the graph with set of vertices V and set of edges $E^C = \{uv \mid uv \notin E\}$.

   Solution: Let $G = (V, E)$ be a graph on at least 6 vertices and v a vertex of G of maximum degree Δ. If $\Delta \geq 3$, then v is the vertex we are looking for. On the other hand, if $\Delta < 3$ then v has degree at least 3 in G^C.

   Now consider any edge uv of G to represent *person u knows person v*. Without loss of generality, consider the case where G has a vertex v of degree 3. Look at the neighbors v_1, v_2, v_3 of v: if any two of them are connected, we get a triangle vv_1v_2 and thus 3 people know each other. If not, we get the triangle $v_1v_2v_3$ in G^C and thus 3 people do not know each other.

9. What is the maximum number of edges in a bipartite graph on n vertices? (Prove your answer.)

   Solution: Let $G = (A \cup B, E)$ be a bipartite graph, with A, B disjoint and $|A| + |B| = n$. Since all the edges of G have one endpoint in A and the other in B, the number of edges $|E|$ of G cannot exceed the number of pairs $(a, b) \in A \times B$, so $|E| \leq |A| \cdot |B| = |A|(n - |A|)$. Intuitively, such a product is maximized when the two factors are equal, so when $|A| = \lfloor n/2 \rfloor$. More formally, we can use the inequality $4xy \leq (x + y)^2$ to get

$$|E| \leq |A|(n - |A|) \leq \frac{(|A| + n - |A|)^2}{4} = \frac{n^2}{4}.$$

   Therefore, the number of edges of a bipartite graph on n edges is at most $n^2/4$.

   Note that $n^2/4$ is exactly the maximum when n is even, because then it is attained by the complete bipartite graph $K_{n/2,n/2}$. When n is odd, the maximum is actually $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil = \frac{n^2-1}{4}$, which is attained by $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.