Discrete Optimization 2024 (EPFL): Problem set of week 12

June 10, 2024

1. Verify that $\{(x, y) \in \mathbb{R}^2 \mid 5x^2 - 2xy + 2y^2 \leq 1\}$ is an ellipse and find its area.

   Solution: We can write $5x^2 - 2xy + 2y^2 = (x + y)^2 + (2x - y)^2$. Therefore, $T(x, y) = (x + y, 2x - y)$ takes the ellipse to the unit disc (of area π). We see from here that T^{-1} takes the unit disc to the ellipse. Therefore, we have an ellipse of area $\pi|\det(T^{-1})| = \pi\frac{1}{3}$.

2. Let B be the unit ball in $\mathbb{R}^3$ and let H be the hyperplane $H = \{x + 3y + 2z = 0\}$.

   Consider the linear transformation

   $$T = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

   What is the (two dimensional) area of $T(B \cap H)$?

   Solution: We find two linearly independent vectors in H: $u = (2, 0, -1)$ and $v = (3, -1, 0)$. We calculate the area of the parallelogram they generate (equal to $\sqrt{|u|^2|v|^2 - < u, v >^2}$) and we get $\sqrt{14}$. Then we consider $T(u) = (0, -3, 4)$ and $T(v) = (2, -3, 5)$ and compute the area of the parallelogram they generate: $\sqrt{109}$. Therefore, the area of $T(B \cap H) = \frac{\sqrt{109}}{\sqrt{14}}\pi$.

3. Let $E \subset \mathbb{R}^3$ be the ellipsoid $E = \{(x, y, z) \mid \frac{(x+2y-z)^2}{4} + \frac{(y-x-z)^2}{9} + \frac{(2x-y-2z)^2}{25} \leq 2\}$. What is the volume of E?

   Solution: We observe that if we consider any point (x, y, z) in E and then apply the linear transformation $T(x, y, z) = ((x + 2y - z)/2, (y -$

$x-z)/3, (2x-y-2z)/5$, then (x,y,z) goes to a point in the unit ball of radius $\sqrt{2}$ whose volume is $(4/3)\pi(sqrt2)^3$. The matrix representing T is the matrix $\begin{pmatrix} 1/2 & 2/2 & -1/2 \\ -1/3 & 1/3 & -1/3 \\ 2/5 & -1/5 & -2/5 \end{pmatrix}$. We calculate its determinant and find it is equal to $-1/3$. It follows now that $1/3$ times the volume of E is equal to $(4/3)\pi(\sqrt{2})^3$. Therefore, the volume of E is euqal to $8\sqrt{2}\pi$.

4. Let $E \subset \mathbb{R}^3$ be the ellipsoid $E = \{(x,y,z) \mid x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1\}$.

   Let H^+ be the half-space $H^+ = \{(x,y,z) \mid x+y+z \geq 0\}$.

   Find an ellipsoid E' such that $E' \supset E \cap H^+$ and $Vol(E') \leq Vol(E)e^{-1/(2(3+1))}$.

   Solution: The linear transformation $T(x,y,z) = (x, \frac{y}{2}, \frac{z}{3})$ takes the ellipsoid E to the unit ball B. Notice that T takes H^+ to the half-space $G^+ = \{x+2y+3z \geq 0\}$. Now we find a linear transformation R that will rotate the space about the origin such that B will remain fixed but G^+ will go to the half space $\{x \geq 0\}$. This part is technical. One elegant way to do this is to take the vector $u = \frac{1}{\sqrt{14}}(1,2,3)$ and complete it to an orthonormal basis: take also the vector $v = \frac{1}{\sqrt{5}}(-2,1,0)$ and the vector w perpendicular to both u and v (you need to find it). Then consider the linear transformation R that takes u, v, w to $(1,0,0), (0,1,0)$, and $(0,0,1)$. This is the inverse of the matrix whose columns are u, v, w.

   The linear transformation $R(T(\cdot))$ takes $E \cap H^+$ to the half ball $B \cap \{x \geq 0\}$. By the half-ball theorem that we proved in class, the ellipsoid $F = \{\frac{16}{9}(x - \frac{1}{4})^2 + \frac{8}{9}y^2 + \frac{8}{9}z^2 \leq 1\}$ contains the half ball $B \cap \{x \geq 0\}$ and we also have $Vol(F) \leq e^{\frac{-1}{2(3+1)}} Vol(B)$.

   If we now apply $T^{-1}(R^{-1}(\cdot))$ on F we get the desired ellipsoid E'. One can get from here the algebraic description of E'. Technically, we need to replace x, y, and z in the algebraic description of F by $R(T(x,y,z))$.