Discrete Optimization 2024 (EPFL): Problem set of week 13

May 31, 2024

1. Consider the polyhedron P defined by $Ax \leq b$ for

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & -2 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}$$

   and $b = (1, 2, 3, 4, 5)$.

   Find a hyperplane (weakly) separating P and $x = (1, 2, 3)$.

   Solution: Calculating Ax we find $Ax = (13, 5, -2, 2, 9)$. We see that Ax does not satisfy $Ax \leq b$ because of the first, second, and fifth coordinates.

   We can therefore take any of the first, second, or fifth constraints and each of them will separate x from P. For example we can take the hyperplane $H = \{2x + 2y + z = 5\}$.

2. Find the volume of the cross polytope, the polytope with the $2n$ vertices in $\mathbb{R}^n$ that are $\pm e_1, \ldots, \pm e_n$.

   Solution: The answer is $\frac{2^n}{n!}$. The reason is that the cross-polytope is the disjoint union of 2^n pyramids generated by the origin and $\pm e_1, \ldots, \pm e_n$.

3. Let P be the simplex with $n+1$ integer vertices in $\mathbb{R}^n$. Assume that P contains an integer interior point. Prove that the volume of P is at least $\frac{n+1}{n!}$.

   Solution: We consider each of the $n+1$ simplices generated by the interior point and one of the $(n-1)$-dimensional faces of the tetrahedron.

The volume of each one is at least $\frac{1}{n!}$ (why?). These $n+1$ simplices are disjoint.

4. Let T be a non-singular linear transformation. Assume that the entries in the matrix representing T are at most D in absolute value and they are all integers. By at most how much can T shrink the size of a vector? That is, how small can be $\frac{|Tx|}{|x|}$? Give a meaningful lower bound.

   Solution: If M is a non-singular linear transformation represented by a matrix whose entries are all smaller than D, then M cannot increase the size of a vector by a factor that is larger than nD. Because each coordinate of Mx cannot be more than nD times the maximum coordinate of x in absolute value. We claim that T cannot shrink the size of any vector by a factor that is larger than $n!D^{n-1}$. This is because each of the entries of the matrix representing T^{-1} is bounded from above by $(n-1)!D^{n-1}$ (Why?). Hence T^{-1} cannot increase the size of a vector by a factor more than $n(n-1)!D^{n-1} = n!D^{n-1}$. This is equivalent to saying that T cannot shrink the size of a vector by a factor larger than $n!D^{n-1}$. That is, for every v (different from 0) we have $\frac{|Tx|}{|x|} \geq \frac{1}{n!D^{n-1}}$.