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Saturated fusion systems as stable retracts of groups

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Outline

1. Bisets as stable maps
2. Fusion systems and idempotents
3. An application to HKR character theory

Notes on the blackboard are in red.

Bisets I

Let S, T be finite p-groups. An (S, T)-biset is a finite set equipped with a left action of S and a free right action of T, such that the actions commute.

Transitive bisets: $[Q, \psi]_S^T := S \times T/(sq, t) \sim (s, \psi(q)t)$ for $Q \leq S$ and $\psi\colon Q \to T$. Q and ψ are determined up to preconjugation in S and postconjugation in T.

(S, T)-bisets form an abelian monoid with disjoint union. The group completion is the *Burnside biset module* $A(S, T)$, consisting of *"virtual bisets"*, i.e. formal differences of bisets.

The $[Q, \psi]$ form a $\mathbb{Z}$-basis for $A(S, T)$.

Bisets II

We can compose bisets $\odot \colon A(R,S) \times A(S,T) \to A(R,T)$ given by $X \odot Y := X \times_S Y$ when X, Y are actual bisets.

$A(S,S)$ is the *double Burnside ring* of S.

Example for D_8 with subgroup diagram. With V_1 as one of the Klein four groups, Q_1 as a reflection contained in V_1, and Z as the centre/half-rotation of D_8, we for example have $[V_1, id] - 2[Q_1, Q_1 \to Z]$ as an element of $A(D_8, D_8)$.

Spectra

Properties that we need about the homotopy category of spectra:

- Suspension spectra $\Sigma^\infty \colon Top_* \to Sp$, where $\Sigma^\infty X$ has the same cohomology as X.
- It's where cohomology theories live.
- It's additive, so we have direct sums (and summands) of spectra.

and

Theorem (Segal conjecture. Carlsson, . . .)

For p-groups S, T:

$$[\Sigma^\infty_+ BS, \Sigma^\infty_+ BT] \approx A(S,T)^\wedge_p \cong \{X \in A(S,T)^\wedge_p \mid |X|/|T| \in \mathbb{Z}\}.$$

Fusion systems I

A fusion system over a finite p-group S is a category $\mathcal{F}$ where the objects are the subgroups $P \leq S$ and the morphisms satisfy:

- $\mathrm{Hom}_S(P, Q) \subseteq \mathcal{F}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$.
- Every $\varphi \in \mathcal{F}(P, Q)$ factors in $\mathcal{F}$ as an isomorphism $P \to \varphi P$ followed by an inclusion $\varphi P \hookrightarrow Q$.

A *saturated* fusion system satisfies a few additional axioms that play the role of Sylow's theorems.

The canonical example of a saturated fusion system is $\mathcal{F}_S(G)$ defined for $S \in \mathrm{Syl}_p(G)$ with morphisms

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) := \mathrm{Hom}_G(P, Q).$$

for $P, Q \leq S$.

Fusion systems II

Example for $D_8 \leq \Sigma_4$: If V_1 consists of the double transpositions in Σ_4, then the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(\Sigma_4)$ gains an automorphism α of V_1 of order 3, and $Q_1 \leq V_1$ becomes conjugate in $\mathcal{F}$ to $Z \leq V_1$.

Characteristic bisets I

If G induces a fusion system on S, we can ask what properties G has as an (S,S)-biset in relation to $\mathcal{F}_S(G)$. Linckelmann-Webb wrote down the essential properties as the following definition:

An element $\Omega \in A(S,S)^{\wedge}_{p}$ is said to be $\mathcal{F}$-*characteristic* if

- Ω is left $\mathcal{F}$-stable: $\operatorname{res}_{\varphi}\Omega = \operatorname{res}_P \Omega$ in $A(P,S)^{\wedge}_{p}$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.
- Ω is right $\mathcal{F}$-stable.
- Ω is a linear combination of transitive bisets $[Q,\psi]_S^S$ with $\psi \in \mathcal{F}(Q,S)$.
- $|\Omega|/|S|$ is invertible in $\mathbb{Z}_{(p)}$.

Characteristic bisets II

G, as an (S, S)-biset, is $\mathcal{F}_S(G)$-characteristic.

Σ_4 as a (D_8, D_8)-biset is isomorphic to

$$\Sigma_4 \cong [D_8, id] + [V_1, \alpha].$$

This biset is $\mathcal{F}_{D_8}(\Sigma_4)$-characteristic. On the other hand, the previous example $[V_1, id] - 2[Q_1, Q_1 \to Z]$ is generated by elements $[Q, \psi]$ with $\psi \in \mathcal{F}$, but it is not $\mathcal{F}$-stable and hence not characteristic.

We prefer a characteristic element that is idempotent in $A(S, S)^{\wedge}_p$.

Characteristic bisets III

Theorem (Ragnarsson-Stancu)

Every saturated fusion system $\mathcal{F}$ has a unique $\mathcal{F}$-characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$, and $\omega_{\mathcal{F}}$ determines $\mathcal{F}$.

For the fusion system $\mathcal{F} = \mathcal{F}_{D_8}(\Sigma_4)$, the characteristic idempotent takes the form

$$\omega_{\mathcal{F}} = [D_8, id] + \frac{1}{3}[V_1, \alpha] - \frac{1}{3}[V_1, id].$$

$B\mathcal{F}$ as a stable retract of BS

The characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)^{\wedge}_{p}$ for a saturated fusion system $\mathcal{F}$ defines an idempotent selfmap

$$\Sigma^{\infty}_{+}BS \xrightarrow{\omega_{\mathcal{F}}} \Sigma^{\infty}_{+}BS.$$

This splits off a direct summand W of $\Sigma^{\infty}_{+}BS$, with properties:

- If $\mathcal{F} = \mathcal{F}_S(G)$, then $W \simeq \Sigma^{\infty}_{+}(BG^{\wedge}_{p})$.
- Each $\mathcal{F}$ has a "classifying space" $B\mathcal{F}$, and $W \simeq \Sigma^{\infty}_{+}B\mathcal{F}$.
- Have maps $i\colon \Sigma^{\infty}_{+}BS \to \Sigma^{\infty}_{+}B\mathcal{F}$ and $\mathrm{tr}\colon \Sigma^{\infty}_{+}B\mathcal{F} \to \Sigma^{\infty}_{+}BS$ s.t. $i \circ \mathrm{tr} = id_{\Sigma^{\infty}_{+}B\mathcal{F}}$ and $\mathrm{tr} \circ i = \omega_{\mathcal{F}}$.

$B\mathcal{F}$ as a stable retract of BS

Each saturated fusion system $\mathcal{F}$ over a p-group S corresponds to the retract $\Sigma^\infty_+ B\mathcal{F}$ of $\Sigma^\infty_+ BS$.

Strategy

- Consider known results for finite p-groups.
- Apply $\omega_{\mathcal{F}}$ everywhere.
- Get theorems for saturated fusion systems, and p-completed classifying spaces.

HKR character theory for fusion systems

(joint with Tomer Schlank & Nat Stapleton)

Hopkins-Kuhn-Ravenel constructed a generalization of group characters in Morava E-theory:

$$\chi_n \colon E_n^*(BG) \to Cl_{n,p}(G; L(E_n^*)).$$

$L(E_n^*)$ is a certain algebra over E_n^*.

$Cl_{n,p}(G; L(E_n^*))$ contains functions valued in $L(E_n^*)$ defined on G-conjugacy classes of n-tuples of commuting elements in G of p-power order.

Theorem (H-K-R)

$$L(E_n^*) \otimes_{E_n^*} E_n^*(BG) \xrightarrow{\simeq} Cl_{n,p}(G; L(E_n^*)).$$

We consider the HKR isomorphism for p-groups

$$L(E_n^*) \otimes_{E_n^*} E_n^*(BS) \xrightarrow{\simeq} Cl_n(S; L(E_n^*)).$$

We try to make $\omega_{\mathcal{F}}$ act on both sides in a way that commutes with the character map. As a result we get:

Pretheorem (R.-Schlank-Stapleton)

For every saturated fusion system $\mathcal{F}$ we have

$$L(E_n^*) \otimes_{E_n^*} E_n^*(B\mathcal{F}) \xrightarrow{\simeq} Cl_{n,p}(\mathcal{F}; L(E_n^*)).$$

For $\mathcal{F} = \mathcal{F}_S(G)$ this recovers the theorem of H-K-R.

Part of proof I

The HKR character map is based on the evaluation map

$$L^n BS \times B(\mathbb{Z}/p^k)^n \xrightarrow{eval} BS$$

for $k \gg 0$.

The free loop space decomposes as

$$L^n BS \simeq \coprod_{\substack{\text{Commuting } n\text{-tuples } \underline{a} \text{ in } S \\ \text{up to } S\text{-conjugation}}} BC_S(\underline{a}),$$

and the evaluation map can be described algebraically as $C_S(\underline{a}) \times (\mathbb{Z}/p^k)^n \to S$ given by

$$(z, t_1, \ldots, t_n) \mapsto z \cdot (a_1)^{t_1} \cdots (a_n)^{t_n}.$$

Part of proof II

Pretheorem (R.-S.-S.)

The characteristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_p^\wedge$ *lifts to a map (actually a matrix of bisets)* M *such that*

$$\begin{array}{ccc} L^n BS \times B(\mathbb{Z}/p^k)^n & \xrightarrow{eval} & BS \\ \downarrow M & & \downarrow \omega_{\mathcal{F}} \\ L^n BS \times B(\mathbb{Z}/p^k)^n & \xrightarrow{eval} & BS \end{array}$$

$\dashrightarrow$: *Stable maps*

Bonus! (Not in the talk)

It is impossible for the lift M to have the form $(?) \times id_{(\mathbb{Z}/p^k)^n}$ and still commute with the evaluation maps. Hence the cyclic factor needs to be used nontrivially.

However the lift M that we construct still commutes with the projection

$$\begin{array}{ccc} L^n BS \times B(\mathbb{Z}/p^k)^n & & \\ M \downarrow & \overset{\pi}{\searrow} & \\ & & B(\mathbb{Z}/p^k)^n \\ & \overset{\pi}{\nearrow} & \\ L^n BS \times B(\mathbb{Z}/p^k)^n & & \end{array}$$

Thank you!

References

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