AVI as a mechanical tool for studying thin-shells based on Kirchhoff-Love constraints

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Objectives

Thin-shell and rod theory using discrete mechanics applied to structures in civil engineering. The aim is to apply structure preserving algorithms to concrete problems in construction. The major objectives of this interdisciplinary work is the search and the development of a practical tool to study irregular surfaces.

Notation and Definitions

In this paper we shall regard a body $\mathcal{B} \subseteq \mathbb{R}^3$ as a smooth orientable Riemannian manifold endowed with a Riemannian metric $\mathbf{G}$. The space $\mathcal{S} \subseteq \mathbb{R}^3$ in which the body moves is also taken to be a smooth orientable Riemannian manifold with a metric $\mathbf{g}$. A *configuration* $\phi : \mathcal{B} \to \mathcal{S}$ is, by definition, an orientation preserving diffeomorphism between $\mathcal{B}$ and its embedded image $\phi(\mathcal{B}) \subseteq \mathcal{S}$. The *configuration space* is defined to be $\mathcal{C} := \{\phi : B \to S \mid \phi \text{ a } C^\infty \text{ embedding}\}$. The deformed body $\phi(\mathcal{B})$ inherits the Riemannian structure of $\mathcal{S}$. We shall call $\mathcal{B}$ the *reference configuration* and $\mathcal{S}$ the *ambient space*. Let $T\mathcal{B}$, $T\mathcal{S}$ be the tangent bundles of $\mathcal{B}$ and $\mathcal{S}$, respectively, and let $T^*\mathcal{B}$, $T^*\mathcal{S}$ be their cotangent bundles. Let $\{X^I\}$ denote the Euclidean coordinates of a point $X \in \mathcal{B}$ relative to the standard basis $\{\widehat{\mathbf{I}}_I\}$ of $\mathbb{R}^3$. Similarly, $\{x^i\}$ are the Euclidean coordinates of a point $x \in \mathcal{S}$ relative to the standard basis $\{\widehat{\mathbf{i}}_i\}$ of $\mathbb{R}^3$.
If $\{\theta^i\}$ is an arbitrary coordinate system on $\mathcal{S}$ we write the coordinate change as a C^∞ map

$$\left(x^1, x^2, x^3\right) \longmapsto \left(\theta^1\left(x^1, x^2, x^3\right), \theta^2\left(x^1, x^2, x^3\right), \theta^3\left(x^1, x^2, x^3\right)\right),$$

with C^∞ inverse. Similarly

$$(X^1, X^2, X^3) \mapsto \left(\Theta^1\left(X^1, X^2, X^3\right), \Theta^2\left(X^1, X^2, X^3\right), \Theta^3\left(X^1, X^2, X^3\right)\right)$$

denotes a coordinate change to an arbitrary coordinate system $\{\Theta^I\}$ in the body. Therefore, the coordinate bases $\{\mathbf{E}_I(\Theta)\}$, $\{\mathbf{e}_i(\theta)\}$ associated to coordinate systems $\{\Theta^I\}$ and $\{\theta^i\}$ are defined, respectively, by

$$\mathbf{E}_I = \frac{\partial X^J}{\partial \Theta^I}\widehat{\mathbf{I}}_J, \qquad \mathbf{e}_i = \frac{\partial x^j}{\partial \theta^i}\widehat{\mathbf{i}}_j, \qquad I, J, i, j = 1, 2, 3 \tag{1}$$

A *motion* of the body is a curve $t \in \mathbb{R} \mapsto \phi_t \in \mathcal{C}$, where $\phi_t(X) := \phi(X, t) \in \mathcal{S}$ for $t \in \mathbb{R}$ fixed and $\phi_0 = \text{Identity}$. The motion ϕ_t is called *regular*[a] if each $\phi_t(\mathcal{B})$ is open and $(\phi_t)^{-1} : \phi_t(\mathcal{B}) \to \mathcal{B}$ exists for all t. Motion of a body occurs due to the action of body forces $\mathbf{b}$ per unit mass and surface traction forces $\mathbf{t}$ per unit area of the boundary $\partial\mathcal{B}$.

Discrete Variational Mechanics

Let $\phi(\mathcal{B}) \times \phi(\mathcal{B})$ be the discrete configuration space associated to the deformed surface $\phi(\mathcal{B})$ and define the discrete path space by $\mathcal{C}_d(\phi(\mathcal{B})) := \{\mathbf{x}_d = \{\mathbf{x}_k\}_{k=0}^N \mid \mathbf{x}_k \in \phi(\mathcal{B}), \mathbf{x}_k = \mathbf{x}_d(t_k), t_k = kh, t_k \in [0, T]\}$; h is the time step. A discrete path $\mathbf{x}_d \in \mathcal{C}_d$ is said to be a solution of the discrete Euler-Lagrange equations if

$$D_2 L_d(\mathbf{x}_{k-1}, \mathbf{x}_k) + D_1 L_d(\mathbf{x}_k, \mathbf{x}_{k+1}) = 0, \quad \text{for all} \quad k = 1, ..., N-1, \tag{2}$$

where $L_d : \phi(\mathcal{B}) \times \phi(\mathcal{B}) \to \mathbb{R}$ is a discrete Lagrangian of order r, that is, it satisfies

$$L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \Delta t) = \int_{t_k}^{t_{k+1}} L(\mathbf{x}, \dot{\mathbf{x}})dt + \mathcal{O}(\Delta t)^{r+1}, \tag{3}$$

where L is the Lagrangian of the continuous systems and $\mathbf{x}(t)$ is the solution of the Euler-Lagrange equations satisfying $\mathbf{x}(t_k) = \mathbf{x}_k$ and $\mathbf{x}(t_{k+1}) = \mathbf{x}_{k+1}$. By applying the discrete Euler-Lagrange equation the points $\{x_k\}$ are iteratively defined by the *one-step integrator* $F_{L_d} : (x_{k-1}, x_k) \mapsto (x_k, x_{k+1})$ which has two important structure preserving properties. First, F_{L_d} is *symplectic*. Second, if L_d is invariant under Lie algebra action, the discrete Lagrangian momentum map J_{L_d} is a conserved quantity: $J_{L_d} \circ F_{L_d} = J_{L_d}$.
In order to achieve conservation of energy we also consider the time interval $[0, T]$ and define the *extended configurations* by $\widetilde{\varphi} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{B}} := \mathbb{R} \times \mathcal{B}$, $\widetilde{\mathcal{S}} := \mathbb{R} \times \mathcal{S}$, and $\mathbb{R}$ is time axis. Thus we get a new condition that ensures conservation of discrete energy:

$$D_3 L_d(x_{k-1}, x_k, h_{k-1}) - D_3 L_d(x_k, x_{k+1}, h_k) = 0, \quad \text{where} \quad h_k = t_{k+1} - t_k \tag{4}$$

Consequently, we get an implicit algorithm giving the value of the time step h_k for each k; the integrator is said to be an *Asynchronous Variational Integrator* (AVI).

[a] In contact problems, where the body may consist of two disconnected components that are brought together during the motion, regularity fails.

Kirchhoff-Love assumptions for thin-shell

According to standard Kirchhoff-Love assumptions, we take the reference shell director[a] $\mathbf{T}$ and the deformed shell director $\mathbf{t}$ to equal the third basis vector respectively

$$\frac{\mathbf{E}_1 \times \mathbf{E}_2}{|\mathbf{E}_1 \times \mathbf{E}_2|} \perp T_{\mathbf{X}}\mathcal{B}, \quad \text{and} \quad \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} \perp T_x\phi(\mathcal{B}) \tag{5}$$

Denote by $\langle \cdot, \cdot \rangle_{\mathbf{x}}$ the standard inner product in $\mathbb{R}^3$ for vectors based at $\mathbf{x} \in \mathcal{S} = \mathbb{R}^3$ and by $\langle \cdot, \cdot \rangle_{\mathbf{X}}$ the standard inner product in $\mathbb{R}^3$ for vectors based at $\mathbf{X} \in \mathcal{B}$. The components $g_{\alpha\beta}$ of the metric tensor on $\phi(\mathcal{B})$ (obtained by pulling back by the inclusion map the inner product $\langle \cdot, \cdot \rangle_{\mathbf{x}}$ on $\mathbb{R}^3$ to $\phi(\mathcal{B})$) are defined by $g_{\alpha\beta}(\mathbf{x}) := \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathbf{x}}$. Similarly define the components $G_{\alpha\beta}$ of the metric on $\mathcal{B}$ by $G_{\alpha\beta}(\mathbf{X}) := \langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle_{\mathbf{X}}$. Let $[G^{\alpha\beta}] := [G_{\alpha\beta}]^{-1}$ and $[g^{\alpha\beta}] := [g_{\alpha\beta}]^{-1}$.

The *strain mesures* relative to the dual spatial surface basis :

$$\epsilon_{ij} := \frac{1}{2}\left(\langle \mathbf{e}_i, \mathbf{e}_j \rangle - \langle \mathbf{E}_i, \mathbf{E}_j \rangle \right)$$

$$\rho_{\alpha\beta} := \left\langle \frac{\partial \mathbf{E}_\alpha}{\partial \theta^\beta}, \mathbf{E}_3 \right\rangle - \left\langle \frac{\partial \mathbf{e}_\alpha}{\partial \theta^\beta}, \mathbf{e}_3 \right\rangle$$

For the simplest properly invariant isotropic constitutive relations we postulate the existence of a *stored energy function* of the displacement field $\mathbf{u}$ of the form

$$W(\mathbf{u}) = \frac{1}{2}\left(\frac{Eh}{1-\nu^2}\right) H^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \frac{1}{2}\left(\frac{Eh^3}{12(1-\nu^2)}\right) H^{\alpha\beta\gamma\delta}\rho_{\alpha\beta}\rho_{\gamma\delta} \tag{6}$$

where E is Young's modulus, ν is Poisson's ratio, h is the thickness of the shell, and

$$H^{\alpha\beta\gamma\delta} = \nu\, G^{\alpha\beta}G^{\gamma\delta} + \frac{1}{2}(1-\nu)\left(G^{\alpha\gamma}G^{\beta\delta} + G^{\alpha\delta}G^{\beta\gamma}\right). \tag{7}$$

To ensure that the bending energy is finite we used biquadratic uniform B-splines.

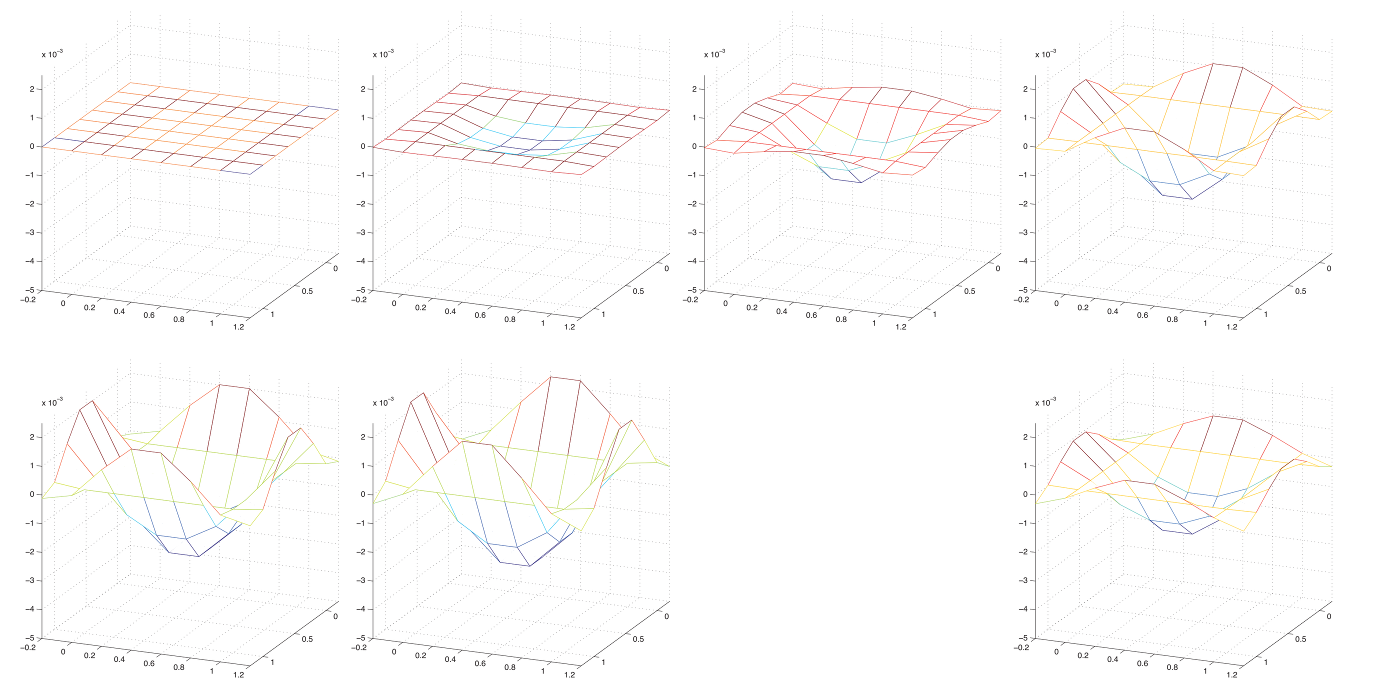
Simulations

To get equilibrium positions, we introduce a dissipative system. And, in the presence of forcing, discrete Noether's theorem exists, which allows us to obtain consistent results.
Let discrete Lagrange d'Alembert principle for discrete mechanical systems with left and right discrete exterior forces F_d^- and F_d^+

$$\delta \sum L_d(q_k, q_{k+1})dt + \sum \left(F_d^-(q_k, q_{k+1})\delta q_k + F_d^+(q_k, q_{k+1}\delta q_{k+1})\right) = 0, \tag{8}$$

And consequently we obtain an integrator $(q_k, q_{k+1}) \mapsto (q_{k+1}, q_{k+2})$, given explicitly by the discrete forced Euler-Lagrange equations for a good discrete Lagrangian L_d

$$D_1 L_d(q_{k+1}, q_{k+2}) + D_2 L_d(q_k, q_{k+1}) + F_d^-(q_{k+1}, q_{k+2}) + F_d^+(q_k, q_{k+1}) = 0 \tag{9}$$



These results were obtained using a module of elasticity $E = 1.1 \cdot 10^9$ and Poisson ratio $\nu = 0.3$, with a plate of lengths $l_1 = l_2 = 1m$, width $h = 0.01m$, and density $\rho = 400kg/m^3$. And we consider this thin-shell as simply supported, using quadratic B-spline instead of classical shape functions.

[a] We assume the existence of a traction vector $\mathbf{t}$ for the motion of $\mathcal{B}$ in $\mathcal{S}$.

Energy behavior

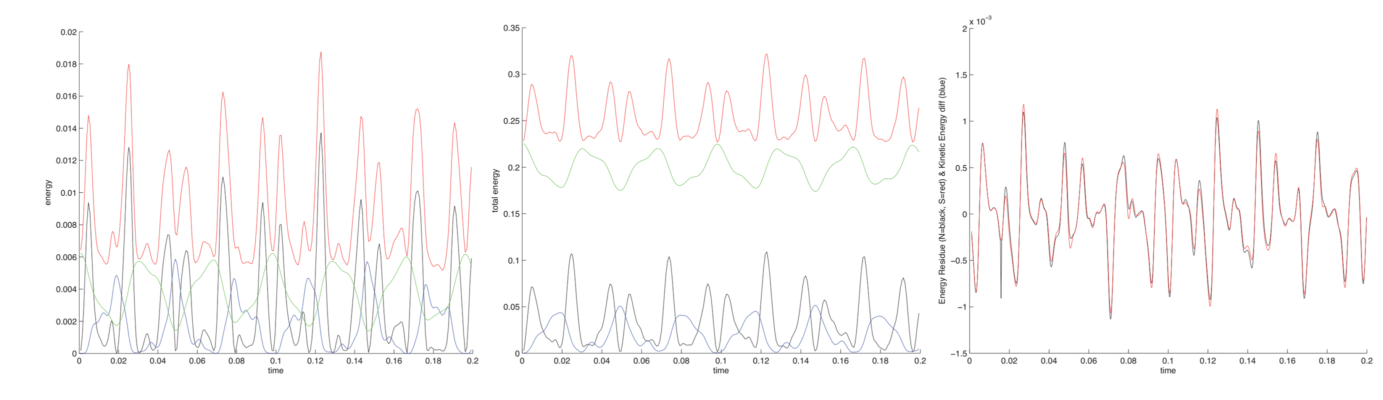
We get consistent and explicit integrator by using discrete Lagrangian L_d, as

$$L_d(\mathrm{x}_k, \mathrm{x}_{k+1}, h) = \frac{h}{2}\left(\frac{\mathrm{x}_{k+1} - \mathrm{x}_k}{h}\right)^T M \left(\frac{\mathrm{x}_{k+1} - \mathrm{x}_k}{h}\right) - hV(\mathrm{x}_k), \tag{10}$$

With L_d, let discrete energy $E_{d,k}$ as previously defined, for time step $h_k = t_{k+1} - t_k$, such that

$$E_{d,k} = -D_3 L_d(\mathrm{x}_k, \mathrm{x}_{k+1}, h) = \frac{1}{2}\left(\frac{\mathrm{x}_{k+1} - \mathrm{x}_k}{h}\right)^T M \left(\frac{\mathrm{x}_{k+1} - \mathrm{x}_k}{h}\right) + V(\mathrm{x}_k) \tag{11}$$

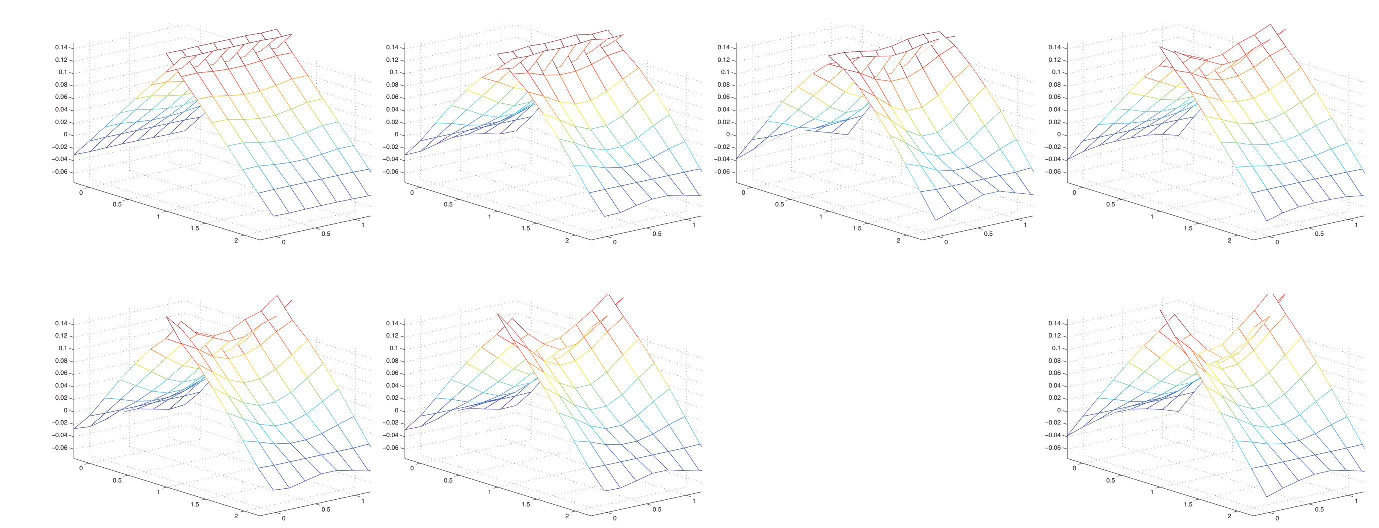
Since $E_{d,k} \neq E_{d,k+1}$ for fixed time-steps, the difference between both sides of the inequality represents the variation of energy between succesive integration steps, called the *energy residue*. And we note that the energy residue is smaller by almost two orders of magnitude in absolute value compared to the energy itself.



Energy behavior for a single element K (left), total energy behavior (middle) of the thin-shell (green = exterior potential energy, blue = elastic potential energy, black = kinetic energy, red = total energy), and energy residue behavior for a single element K (right) at the center (black = residue using kinetic energy on nodes, red = residue using kinetic energy on element K).

One edge and two plates

We consider two thin-shells of same sizes, leaning against each other, so they form an edge. And, as previously, we get equilibrium position, by introducing a dissipative system.



These results were obtained using a module of elasticity $E = 1.1 \cdot 10^9$ and Poisson ratio $\nu = 0.3$, with two plates of lengths $l_1 = l_2 = 1m$, width $h = 0.01m$, and density $\rho = 400kg/m^3$. And we consider this thin-shells as simply supported on the boundaries except on the edge, using quadratic B-spline instead of classical shape functions.

References

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