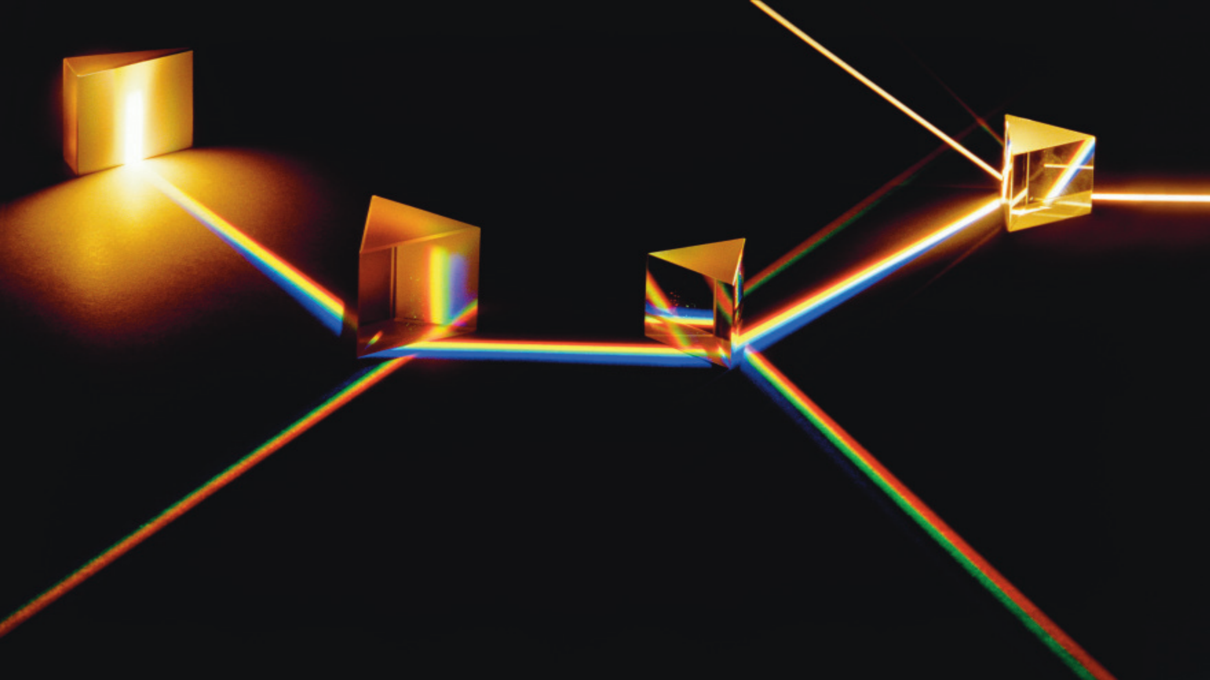
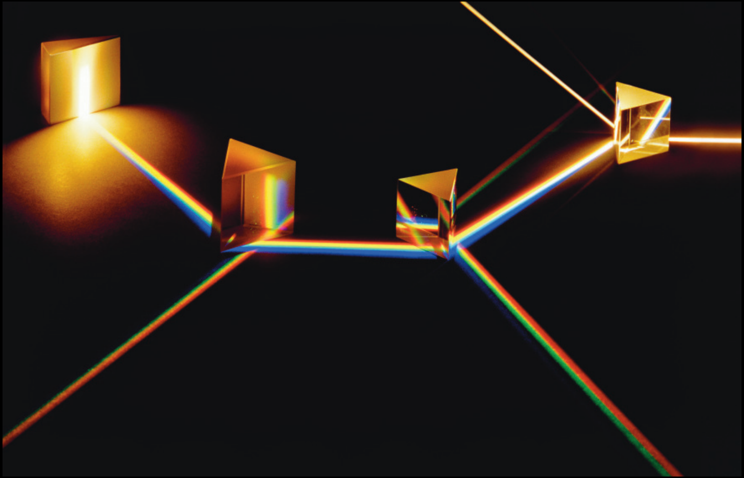




Euclidean Distance Matrices:
A Short Walk Through Theory, Algorithms and Applications

Ivan Dokmanić, Reza Parhizkar, Juri Ranieri and Martin Vetterli

Overview:

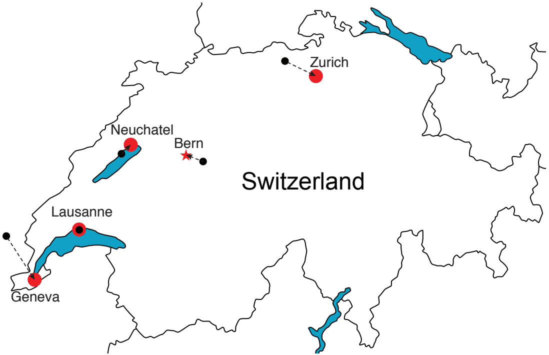
- Motivation
- Euclidean Distance Matrices (EDM) and their properties
- Algorithms for EDMs
- Applications of EDMs
- Conclusions and open problems

Motivation

- We land at the GVA airport and we have lost our map of Switzerland...
- We have only a train schedule!

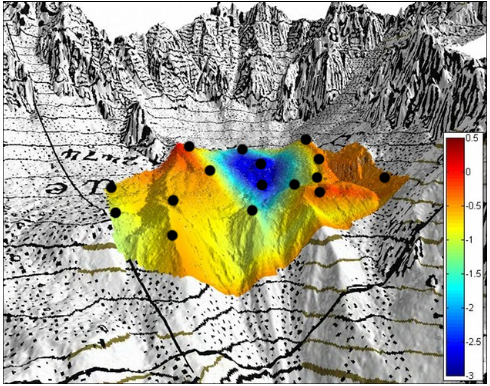
	Lausanne	Geneva	Zürich	Neuchâtel	Bern
Lausanne	0	33	128	40	66
Geneva	33	0	158	64	101
Zürich	128	158	0	88	56
Neuchâtel	40	64	88	0	34
Bern	66	101	56	34	0

Distances in minutes between five swiss cities



Locations (red) and estimated locations (black) of the cities.

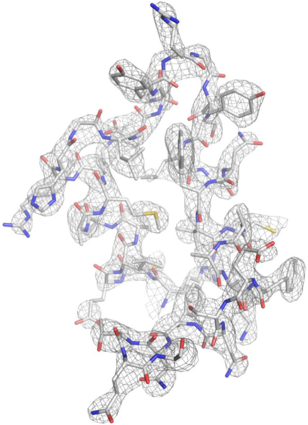
Sensor network localization problem



Sensor network deployed on a rock glacier

- We measure the distances between the sensor nodes,
- The distances are noisy and some are missing,
- How do we reconstruct the locations of the sensors?

Molecular conformation problem

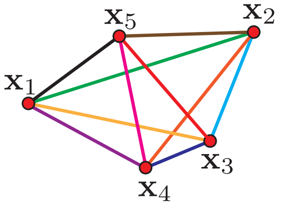


Density map and structure of a molecule [10.7554/elife.01345]

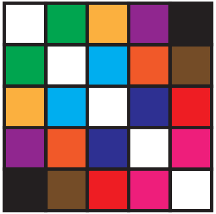
- We measure the distances between the atoms,
- The distances are given as intervals, and some are missing,
- How do we reconstruct the molecule?

Euclidean Distance Matrix

$$\mathbf{X} = \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \ldots & \mathbf{x}_5 \\ | & & | \end{bmatrix}$$



$$\mathrm{edm}(\mathbf{X}) =$$



- Consider a set of n points $\mathbf{X} \in \mathbb{R}^{d \times n}$,
- $\mathrm{edm}(\mathbf{X})$ contains the squared distances between the points,
- Equivalently, $\mathrm{edm}(\mathbf{X}) = \mathbf{1}\,\mathrm{diag}(\mathbf{X}^\top\mathbf{X})^\top - 2\mathbf{X}^\top\mathbf{X} + \mathrm{diag}(\mathbf{X}^\top\mathbf{X})\mathbf{1}^\top$.

EDM properties: rank and essential uniqueness

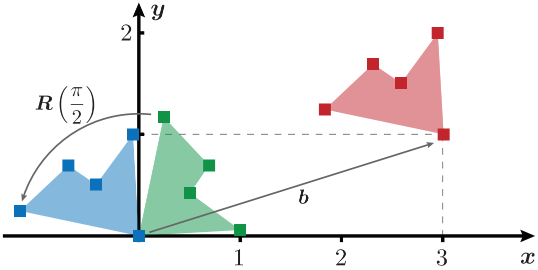
- The rank of an EDM depends only on the dimensionality of the points:

  Theorem (Rank of EDMs)

  Rank of an EDM corresponding to points in $\mathbf{R}^d$ is at most $d+2$.

- EDMs are invariant to rigid transformations (translation, rotations, reflections):

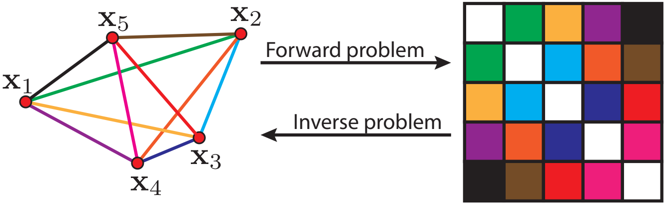
$$\mathrm{edm}(\mathbf{X}) = \mathrm{edm}(\mathbf{RX} + b\mathbf{1}^\top).$$



Counting the degrees of freedom

- Consider n points in $\mathbb{R}^d$, then we have $\#_X = nd$ degrees of freedom (DoF).
- If we describe the points with a symmetric rank-$(d+2)$ matrix, we have $\#_{\text{DOF}} = n(d+1) - \frac{(d+1)(d+2)}{2}$ DoF.
- For large n and fixed d, $\frac{\#_{\text{DOF}}}{\#_X} \sim \frac{d+1}{d}$: the rank property is not a *tight* description of an EDM.

Forward and inverse problem related to EDMs



- Forward problem: given the points $\mathbf{X}$, find the EDM $\text{edm}(\mathbf{X})$.
- Inverse problem: given the distances $\|\mathbf{x}_i - \mathbf{x}_j\|^2$, find the points $\mathbf{X}$.
  - Distances may be *noisy*,
  - Some distances may be *missing*,
  - We may loose the naming of the *distances*.
  - Significantly harder than the forward problem.

Examples of inverse problems

Algorithms: multidimensional scaling

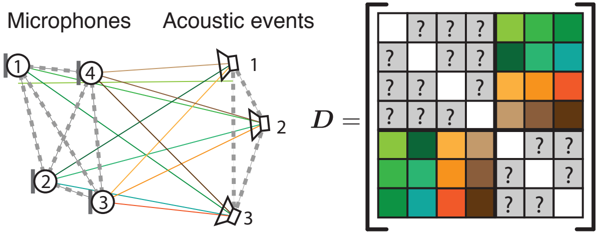
Algorithms: alternating descent of the s-stress function

Other algorithms

Algorithms: performance comparison

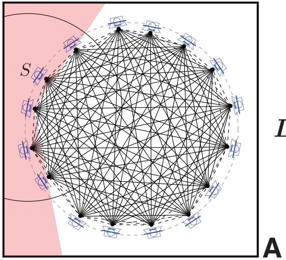
Applications of EDM

Applications: Multidimensional unfolding of echoes

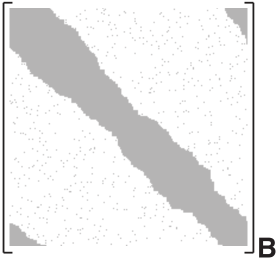


- Multidimensional unfolding:
  - Divide the points in two sets,
  - Measure the distances between the points belonging to different subsets,
  - Other distances are unknown.
- Example: calibration of a microphones with sources at unknown locations.
- Many distances are missing, the measured ones are noisy.

Applications: Calibration in ultrasound tomography



$D =$

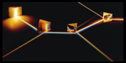


- (A) Ultrasound transceivers deployed on a circle, their location is unknown.
- Each transceiver can measure the distance w.r.t. the transceivers in front.
- (B) We can fill the EDM, with structured missing entries.
- Problem: estimate the locations of the transceiver to improve the imaging performance.

Applications: Can you hear the shape of a room?

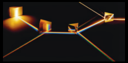
Applications: Sparse Phase Retrieval

Open problems



- Distance matrices on manifolds
- Projections of EDMs on lower dimensional subspaces
- Efficient algorithms for distance labeling
- Analytical local minimum of s-stress

Conclusions



- Definition of EDMs
- Properties of EDMs
- Inverse problems with EDMs

Application	missing distances	noisy distances	unlabeled distances
Wireless sensor networks *	✔	✔	×
Molecular conformation	✔	✔	×
Hearing the shape of a room *	×	✔	✔
Indoor localization	×	✔	✔
Calibration	✔	✔	×
Sparse phase retrieval *	×	✔	✔

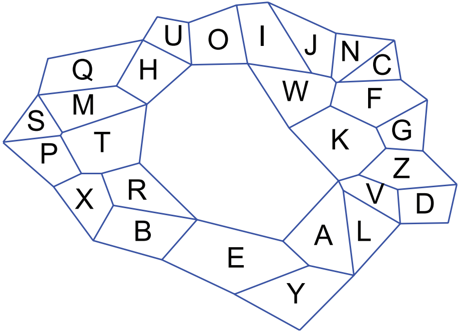
Literature

Acknowledgments

- Farid Movahedi Naini for helping with the numerical experiments.
- Funding resources:
  - Swiss National Science Foundation, Grant 200021 138081
  - European Research Council SPARSAM, Grant 247006
  - Google ?

Thanks for the attention!

QWERTY Keyboard



EDM Keyboard