A class of non-coercive problems in electromagnetism and linear elasticity

We consider the problem: find $u \in X$ such that

$$Au = f \in X', \tag{1}$$

where X is a separable Hilbert space and $A : X \to X'$ is a bounded linear operator. We assume that A is a compact perturbation of a mapping $A_0 : X \to X'$ satisfying

$$\exists \alpha > 0; \quad |\langle A_0(v+w), v-w \rangle_{X' \times X}| \ \geq \ \alpha \, \|u\|_X^2 \quad \forall u = v + w \in X = V \oplus W,$$

on a stable direct decomposition $V \oplus W$ of X. We recall that such an operator A is Fredholm of index zero [1] and provide conditions ensuring the convergence of Galerkin schemes based on a variational formulation of (1).

This abstract theory is useful when studying variational problems whose energy space is $X = H(\mathbf{curl}, \Omega)$ (cf. [1] and the references therein) or $X = H(\mathrm{div}, \Omega)$. Indeed, as the canonical injection from the former two spaces into $[L^2(\Omega)]^3$ is not compact, the differential operator A does not satisfy a Gårding inequality when its term of order zero has the wrong sign. We illustrate such a situation in the context of Maxwell equations [3] and in the case of mixed formulations for linear elasticity [2, 4, 5]. In each example, we show that a judicious decomposition of X renders suitable the application of a Fredholm alternative and simplifies the convergence analysis of the finite element discretization of the problem.

References

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