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REDUCED BASIS METHOD FOR PARAMETRIZED ELLIPTIC OPTIMAL CONTROL PROBLEMS*

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Abstract. We propose a suitable model reduction paradigm – the certified reduced basis method (RB) – for the rapid and reliable solution of parametrized optimal control problems governed by partial differential equations (PDEs). In particular, we develop the methodology for parametrized quadratic optimization problems with elliptic equations as constraint. Firstly, we recast the optimal control problem in the framework of saddle-point problems in order to take advantage of the already developed RB theory for Stokes-type problems. Then, the usual ingredients of the RB methodology are provided: a Galerkin projection onto a low-dimensional space of basis functions properly selected by an adaptive procedure; an affine parametric dependence enabling to perform competitive Offline-Online splitting in the computational procedure; an efficient and rigorous a posteriori error estimate on the state, control and adjoint variables as well as on the cost functional. Finally, the reduction scheme is applied to some numerical tests confirming the theoretical results and showing the efficiency of the proposed technique.

Key words. reduced basis methods, parametrized optimal control problems, saddle-point problems, model order reduction, PDE-constrained optimization, a posteriori error estimate.

AMS subject classifications. 49J20, 65K10, 65M15, 65M60, 65N12, 93C20

1. Introduction. The numerical solution of PDE-constrained optimization problems is usually computationally demanding, since it requires the solution of a system of PDEs arising from the optimality conditions – the state problem, the adjoint problem and a further set of equations ensuring the optimality of the solution. This task becomes even more challenging whenever the state system (or the cost functional to be minimized) depend on a set of parameters – which can specify physical or geometrical properties of interest – and we are interested to solve an optimal control problem for many different scenarios corresponding to different sets of parameter values. In this case, standard techniques built over full-order discretization methods such as the finite element method may yield an overwhelming computational complexity. Therefore, when performing the optimization process for many different parameter values (*many-query* context) or when, for a given new configuration, we need to compute the solution in a rapid way (*real-time* context), the computational effort may be unacceptably high and, often, unaffordable. Substantial computational saving becomes possible thanks to a *reduced order model* (ROM) which relies on the reduced basis (RB) method [30, 25], which allows to solve a parametrized PDE problem for any new value of the parameters (inexpensive *Online* evaluation) once a set of (full-order) solutions have been computed for selected values of the parameter set and stored (expensive *Offline* database construction).

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We denote with $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$ a p-vector of parameters representing either physical and/or geometrical quantities of interest, while y represents the state variable, u the control variable, $\mathcal{J}$ the objective functional, and $\mathcal{E}(\cdot, \cdot; \boldsymbol{\mu})$ the residual of the state equation. The general form of a parametrized optimal control problem reads: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$(OCP_{\boldsymbol{\mu}}) \qquad \min_{y,u} \mathcal{J}(y, u; \boldsymbol{\mu}) \qquad \text{subject to} \qquad \mathcal{E}(y, u; \boldsymbol{\mu}) = 0.$$

In this work we bound our analysis to the most typical linear/quadratic case, i.e. to optimal control problems featuring quadratic cost functionals and linear (scalar coercive) elliptic PDEs as constraint.
From an abstract point of view, the mapping $\boldsymbol{\mu} \mapsto (y(\boldsymbol{\mu}), u(\boldsymbol{\mu}))$ defines a *smooth* and rather *low-dimensional* parametrically induced manifold $\mathcal{M} = \{(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in X : \boldsymbol{\mu} \in \mathcal{D}\}$, where $y(\boldsymbol{\mu})$ and $u(\boldsymbol{\mu})$ are the state and control solutions of $(OCP_{\boldsymbol{\mu}})$ and X is a suitable functional space. In a classical discretization approach, after introducing an approximation space $X^{\mathcal{N}}$ of (typically very large) dimension $\mathcal{N}$ – e.g. a finite element (FE) space – for every value of the parameters $\boldsymbol{\mu}$ we are supposed to solve the whole optimal control problem in order to compute the solution $(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu}))$, ignoring the possibly *smooth* relation between parameters and solutions. A reduced (basis) approach is premised e.g. upon a classical FE method and consists in a low-order approximation of the *truth* manifold $\mathcal{M}^{\mathcal{N}}$, based on *(i)* computation of some snapshots of the *truth* manifold $\mathcal{M}^{\mathcal{N}}$, and *(ii)* a Galerkin projection onto the space spanned by the precomputed snapshots.
The main ingredients of the reduced basis (RB) methods [25, 30] are the following ones: (i) a rapidly convergent global approximation (Galerkin projection) onto a space spanned by solution of the original problem at some selected parameters value; (ii) a rigorous a posteriori error estimation procedures which provides inexpensive yet sharp bounds for the error between the RB and the *truth* solution; (iii) an Offline/Online computational procedure, i.e. an efficient splitting between a time-consuming and parameter independent Offline stage and an inexpensive Online calculation for each new input/output evaluation.

Computational reduction strategies such as RB methods or proper orthogonal decomposition (POD) have already been employed to speedup the solution of optimal control, as well as other PDE-constrained optimization problems. First examples of optimal control problems solved by exploiting computational reduction techniques have been addressed by Ito and Ravindran, in the context either of (a preliminary version of) the RB method [18] or of the proper orthogonal decomposition method [27]. Other recent works dealing with optimal control problems through POD techniques have been addressed for instance by Kunisch and Volkwein [20] (and reference therein). More recent contributions dealing with RB methods have been presented in both the elliptic case by Quarteroni, Rozza and Quaini [26], Tonn, Urban and Volkwein [35], Grepl and Kärcher [9], and the parabolic case by Dedè [5, 6]. However, in all these works the control variable is low-dimensional, e.g. a set of real numbers that could be treated themselves as parameters. We aim at developing a certified reduced framework that enables to handle infinite dimensional (either distributed and/or boundary) control functions. In this context, designing a strategy for the reduction of the complexity of the optimal control problem (that is treated as a whole, with respect to all its variables simultaneously) becomes mandatory.
Furthermore, an efficient and rigorous a posteriori error estimation, necessary both for constructing the reduced order model and for measuring its accuracy, is still missing

for a large class of optimal control problems. For example, the a posteriori estimators for the error in the cost functional and in the control variable proposed in some previous works [5, 6] are efficient in practice but unfortunately lack of rigorousness, whereas the estimator proposed in [35] is proved to be rigorous but not efficient. Only recently an efficient and rigorous estimator has been proposed in the case of constant control function in [9]. In this work we aim at developing both efficient and rigorous a posteriori error bounds in order to estimate, simultaneously, the errors on the optimal control, the state variable and the cost functional.

With reference to the basic ingredients of the RB method previously introduced, we point out that:

(i) in our approach the reduced scheme is built directly over the optimality conditions system rather than on the original optimization problem, following an *optimize-then-discretize-then-reduce* approach. Indeed, we first derive the optimality system (*optimize* step), then we introduce its *truth* finite element (FE) approximation (*discretize* step) and finally we provide the RB approximation for the whole optimality system (*reduce* step).

(ii) the reduced basis is made of optimal solutions of the original problem, hence the computation of each basis function requires the resolution of the FE *truth* approximation; moreover, the reduced spaces are built for both the state, control and adjoint variables.

(iii) to ensure the well-posedness of the RB approximation, and in order to provide an *a posteriori* error estimate for the optimal control problem, we take advantage of the RB theory developed for Stokes-type problems [22, 29, 32] by recasting the optimal control problem in the framework of saddle-point problems;

(iv) we rely on the the affine parameter dependence assumption, which provides the possibility to extract the parameter dependent components from our operators and thus exploit an Offline/Online computational procedure.

The paper is structured as follows. In §2 we introduce the formulation of parametrized linear/quadratic optimal control problems governed by elliptic coercive PDEs with affine parameter dependence; after having recast the problem in the framework of saddle-point problems, we briefly discuss its FE *truth* approximation, recalling the necessary assumptions to ensure the well-posedness. In §3 we discuss the RB approximation and the main features of the method, focusing on the corresponding stability condition for the RB approximation. Then in §4 we deal with the a posteriori error estimation for the RB solution and functional based on the *Babuška stability theory* [2]. Finally, in §5 some numerical examples are presented.

2. Parametrized optimal control problems.

In this section we introduce the parametrized optimal control problems we focus on and, once recast in the framework of saddle-point problems, we prove a well-posedeness result. Finally we introduce the *truth* FE approximation.

2.1. Problem definition.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be an open and bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, and $\mathcal{D} \subset \mathbb{R}^p$ be a prescribed p-dimensional compact set of parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$, with $p \geq 1$. Let Y, U be two Hilbert spaces[1] for the state and control variables y and u respectively, while the Hilbert

[1] Typically the state space Y is a closed subspace of $H^1(\Omega)$ such that $H^1_0(\Omega) \subset Y \subset H^1(\Omega)$, while the control space can be given for example by $U = L^2(\omega)$, being ω a portion of the domain or of the boundary. We do not treat here the case of control-constrained problems, i.e. problems where the control space is a closed and convex set in a Hilbert space rather than a Hilbert space itself.

space $\mathcal{Z} \supset Y$ shall denote the observation space. Given another Hilbert space Q, we define the linear *constraint equation* in the form

$$a(y, q; \boldsymbol{\mu}) = c(u, q; \boldsymbol{\mu}) + \langle G(\boldsymbol{\mu}), q\rangle \qquad \forall q \in Q, \tag{2.1}$$

where the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu}) : Y \times Q \to \mathbb{R}$ represents a linear elliptic operator, the bilinear form $c(\cdot,\cdot;\boldsymbol{\mu}) : U \times Q \to \mathbb{R}$ expresses the action of the control and $G(\boldsymbol{\mu}) \in Q'$ is a linear continuous functional acting as a forcing term. The quadratic *cost functional* to be minimized is given by

$$J(y, u; \boldsymbol{\mu}) = \frac{1}{2} m(y - y_d(\boldsymbol{\mu}), y - y_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(u, u; \boldsymbol{\mu}), \tag{2.2}$$

where $\alpha > 0$ is a given constant, $y_d(\mathbf{x}, \boldsymbol{\mu}) \in \mathcal{Z}$ is a given parameter-dependent observation function, the bilinear form $m(\cdot,\cdot;\boldsymbol{\mu}) : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ defines the objective of the minimization while the bilinear form $n(\cdot,\cdot;\boldsymbol{\mu}) : U \times U \to \mathbb{R}$ acts as a penalization term for the control variable. The parametrized optimal control problem reads: for any given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{y,u} J(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad (y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in Y \times U \text{ solves (2.1)}. \tag{2.3}$$

Let us specify the assumptions on the linear and bilinear forms introduced above. We firstly remark that, since we are interested in considering second-order coercive elliptic equation as constraint, we can assume without loss of generality that $Q \equiv Y$[2]. Then, we assume that the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$ is bounded and coercive over Y for any $\boldsymbol{\mu} \in \mathcal{D}$, i.e. there exists a constant $\tilde{\alpha}_0 > 0$ such that

$$\tilde{\alpha}(\boldsymbol{\mu}) = \inf_{z \in Y} \frac{a(z, z; \boldsymbol{\mu})}{\|z\|_Y^2} \geq \tilde{\alpha}_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{2.4}$$

We assume that the bilinear form $c(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric and bounded, and the bilinear form $n(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric, bounded and coercive. Moreover, we assume the bilinear form $m(\cdot,\cdot;\boldsymbol{\mu})$ to be symmetric, continuous and positive in the norm induced by the space $\mathcal{Z}$. Holding these assumptions, the existence of a unique solution $(y, u) \in Y \times U$ of the optimal control problem (2.3) can be easily proved by applying either Lions theory [21] or Lagrange multiplier theory [17, 13]. Here however, in view of the application of the RB method, we are interested in recasting the problem in the framework of saddle-point problems.

Before addressing this issue, let us make an additional assumption, crucial to Offline-Online procedures, by assuming the bilinear and linear forms, as well as the observation function, to be affine[3] in the parameter $\boldsymbol{\mu}$, i.e. for some finite $\tilde{Q}_*$, $* \in$

[2]We therefore limit ourselves to consider Galerkin variational problems as state equations rather than Petrov-Galerkin problems. We remark that while at the continuous level it seems useless to keep a different notation for the spaces Y and Q, it will be crucial in order to correctly construct the RB approximation (as well as to generalize the method to the case $Y \neq Q$).

[3]If this assumption does not hold, it could be recovered through the so-called *Empirical Interpolation Method* (EIM); see [31] for an application to optimal control problems.

$\{a, c, n, m, g, d\}$, they can be expressed as

$$
\begin{aligned}
a(z,q;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_a} \tilde{\Theta}_a^q(\boldsymbol{\mu})\, a^q(z,q), & c(v,q;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_c} \tilde{\Theta}_c^q(\boldsymbol{\mu})\, c^q(v,q),\\
m(y,z;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_m} \tilde{\Theta}_m^q(\boldsymbol{\mu})\, m^q(y,z), & n(u,v;\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_n} \tilde{\Theta}_n^q(\boldsymbol{\mu})\, n^q(u,v),\\
\langle G(\boldsymbol{\mu}), q\rangle &= \sum_{q=1}^{\tilde{Q}_g} \tilde{\Theta}_g^q(\boldsymbol{\mu})\, \langle G^q, q\rangle, & y_d(\mathbf{x},\boldsymbol{\mu}) &= \sum_{q=1}^{\tilde{Q}_d} \tilde{\Theta}_d^q(\boldsymbol{\mu})\, y_d^q(\mathbf{x}),
\end{aligned}
\tag{2.5}
$$

for given *smooth* $\boldsymbol{\mu}$-dependent function $\tilde{\Theta}_*^q(\boldsymbol{\mu})$ and continuous $\boldsymbol{\mu}$-independent bilinear and linear forms $a^q(\cdot,\cdot)$, $c^q(\cdot,\cdot)$, $m^q(\cdot,\cdot)$, $n^q(\cdot,\cdot)$, $n^q(\cdot,\cdot)$, G^q and functions $y_d^q \in \mathcal{Z}$.

2.2. Saddle-point formulation. In order to formulate the optimal control problem (2.3) as a saddle-point problem, let us denote with $X = Y \times U$ the product space between the state space Y and the control space U, equipped with the inner product $(\underline{x}, \underline{w})_X = (y,z)_Y + (u,v)_U$ and norm $\|\cdot\|_X = \sqrt{(\cdot,\cdot)_X}$, being $\underline{x} = (y,u) \in X$, $\underline{w} = (z,v) \in X$. Furthermore, we define the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu}) : X \times X \to \mathbb{R}$ as

$$\mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) = m(y,z;\boldsymbol{\mu}) + \alpha n(u,v;\boldsymbol{\mu}), \qquad \forall \underline{x}, \underline{w} \in X,$$

and the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu}) : X \times Q \to \mathbb{R}$ as

$$\mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = a(z,q;\boldsymbol{\mu}) - c(v,q;\boldsymbol{\mu}), \qquad \forall \underline{w} \in X, q \in Q.$$

By defining the linear functional $\underline{F}(\boldsymbol{\mu}) = m(y_d(\boldsymbol{\mu}),\cdot;\boldsymbol{\mu}) \in X'$, we can express the cost functional as $J(y,u;\boldsymbol{\mu}) = \mathcal{J}(\underline{x};\boldsymbol{\mu}) + t(\boldsymbol{\mu})$, where $t(\boldsymbol{\mu}) = \frac{1}{2} m(y_d(\boldsymbol{\mu}), y_d(\boldsymbol{\mu});\boldsymbol{\mu})$ and

$$\mathcal{J}(\underline{x};\boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(\underline{x},\underline{x};\boldsymbol{\mu}) - \langle \underline{F}(\boldsymbol{\mu}), \underline{x}\rangle. \tag{2.6}$$

Since for any fixed $\boldsymbol{\mu} \in \mathcal{D}$ the constant term $t(\boldsymbol{\mu})$ does not affect the minimizer of $J(\cdot,\cdot;\boldsymbol{\mu})$, we can reformulate the problem (2.3) as follows: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{\underline{x}\in X} \mathcal{J}(\underline{x};\boldsymbol{\mu}) \qquad \text{subject to} \qquad \mathcal{B}(\underline{x}, q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle \quad \forall q \in Q. \tag{2.7}$$

It is well known (see for instance [11, 33]) that the constrained optimization problem (2.7) falls into the framework of saddle-point problems, for which the existence and uniqueness of a solution is well-established by Brezzi theorem [4] under the following conditions:

(i) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is continuous over $X \times X$:

$$\gamma_a(\boldsymbol{\mu}) = \sup_{\underline{x}\in X} \sup_{\underline{w}\in X} \frac{\mathcal{A}(\underline{x},\underline{w};\boldsymbol{\mu})}{\|\underline{w}\|_X \|\underline{x}\|_X} < +\infty, \qquad \forall \boldsymbol{\mu} \in \mathcal{D};$$

(ii) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is coercive over $X_0 = \{\underline{w} \in X \colon \mathcal{B}(\underline{w}, q;\boldsymbol{\mu}) = 0 \quad \forall q \in Q\} \subset X$, i.e. there exists a constant $\alpha_0 > 0$ such that

$$\alpha(\boldsymbol{\mu}) = \inf_{\underline{x}\in X_0} \frac{\mathcal{A}(\underline{x},\underline{x};\boldsymbol{\mu})}{\|\underline{x}\|_X^2} \geq \alpha_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D};$$

(iii) the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ is continuous over $X \times Q$

$$\gamma_b(\boldsymbol{\mu}) = \sup_{\underline{w}\in X} \sup_{q\in Q} \frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} < +\infty, \qquad \forall \boldsymbol{\mu} \in \mathcal{D};$$

(iv) the bilinear form $\mathcal{B}(\cdot,\cdot)$ satisfies the inf-sup condition over $X \times Q$, i.e. there exists a constant $\beta_0 > 0$ such that

$$\beta(\boldsymbol{\mu}) = \inf_{q\in Q} \sup_{\underline{w}\in X} \frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}; \tag{2.8}$$

(v) the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ is symmetric and non-negative over X.

Holding these assumptions, the optimal control problem has a unique solution $\underline{x}(\boldsymbol{\mu}) \in X$ for any $\boldsymbol{\mu} \in \mathcal{D}$, and that solution can be determined by solving the following saddle-point problem (i.e. the optimality system): given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$ such that

$$\begin{cases} \mathcal{A}(\underline{x}(\boldsymbol{\mu}),\underline{w};\boldsymbol{\mu}) + \mathcal{B}(\underline{w},p(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}),\underline{w}\rangle & \forall \underline{w} \in X, \\ \mathcal{B}(\underline{x}(\boldsymbol{\mu}),q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}),q\rangle & \forall q \in Q, \end{cases} \tag{2.9}$$

where $p(\boldsymbol{\mu})$ is the Lagrange multiplier (i.e. the adjoint variable) associated to the constraint. In fact, if we introduce the Lagrangian functional $\mathcal{L}(\cdot;\boldsymbol{\mu}) : X \times Q \to \mathbb{R}$

$$\mathcal{L}(\underline{x},p;\boldsymbol{\mu}) = \mathcal{J}(\underline{x},\boldsymbol{\mu}) + \mathcal{B}(\underline{x},p;\boldsymbol{\mu}) - \langle G(\boldsymbol{\mu}),p\rangle, \tag{2.10}$$

the equations in (2.9) are nothing but the first-order necessary (and sufficient[4]) optimality conditions for the unconstrained optimization problem of finding saddle-points $(\underline{x},p) \in X \times Q$ of the Lagrangian, i.e. (2.9) is equivalent to

$$\nabla\mathcal{L}(\underline{x}(\boldsymbol{\mu}),p(\boldsymbol{\mu});\boldsymbol{\mu})[\underline{w},q] = 0, \qquad \forall(\underline{w},q) \in X \times Q. \tag{2.11}$$

Furthermore, we remark that the optimality system (2.9) is in fact the usual optimality system given by the state equation, the adjoint equation and the optimality equation. Let us now verify the fulfillment of the hypotheses (i)-(v).

Lemma 2.1. *The bilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ satisfy the Brezzi assumptions (i)-(v).*

Proof. It is sufficient to exploit the assumptions made on the bilinear forms $a(\cdot,\cdot;\boldsymbol{\mu})$, $c(\cdot,\cdot;\boldsymbol{\mu})$, $m(\cdot,\cdot;\boldsymbol{\mu})$ and $n(\cdot,\cdot;\boldsymbol{\mu})$, see for instance [11]. In view of the design of a suitable RB scheme it is useful to show here the proof of the fulfillment of the inf-sup condition for the bilinear form $\mathcal{B}(\cdot,\cdot)$. We exploit the fact that $Y \equiv Q$ and the coercivity property of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$

$$\sup_{0\neq\underline{w}\in X} \frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X} = \sup_{0\neq(z,v)\in Y\times U} \frac{a(z,q;\boldsymbol{\mu}) - c(v,q;\boldsymbol{\mu})}{(\|z\|_Y^2 + \|v\|_U^2)^{1/2}}$$
$$\underset{(z,v)=(q,0)}{\geq} \frac{a(q,q;\boldsymbol{\mu})}{\|q\|_Y} \geq \tilde{\alpha}(\boldsymbol{\mu})\|q\|_Y = \tilde{\alpha}(\boldsymbol{\mu})\|q\|_Q.$$

Note that the inequality $\beta(\boldsymbol{\mu}) \geq \tilde{\alpha}(\boldsymbol{\mu})$ plays a crucial role in the following. □

[4]We recall that in the linear/quadratic case the usual second order sufficient optimality condition – requiring the second derivative of the Lagrangian functional to be coercive on the null space of the linearized state equation [17, 13] – reduces to the assumption (ii) stated above.

Then, for any $\boldsymbol{\mu} \in \mathcal{D}$, the optimal control problem (2.3) is equivalent to the saddle-point problem (2.9) and the latter admits a unique solution $(\underline{x}(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$. Moreover, the solution satisfies the stability estimate

$$\|\underline{x}(\boldsymbol{\mu})\|_X + \|p(\boldsymbol{\mu})\|_Q \leq C(\|\underline{F}(\boldsymbol{\mu})\|_{X'} + \|G(\boldsymbol{\mu})\|_{Q'}) \qquad \forall \boldsymbol{\mu} \in \mathcal{D},$$

where C is a positive constant (possibly $\boldsymbol{\mu}$-dependent).

Let us finally observe that, thanks to the affine parameter dependence assumption (2.5), an affine decomposition holds also for the bilinear and linear forms in (2.9), i.e. for some finite Q_a, Q_b, Q_f, Q_g, they can be expressed as

$$\mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\, \mathcal{A}^q(\underline{x}, \underline{w}), \qquad \mathcal{B}(\underline{w}, p; \boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\, \mathcal{B}^q(\underline{w}, p) \tag{2.12}$$

$$\langle G(\boldsymbol{\mu}), q\rangle = \sum_{q=1}^{Q_g} \Theta_g^q(\boldsymbol{\mu})\, \langle G^q, q\rangle, \qquad \langle \underline{F}(\boldsymbol{\mu}), \underline{w}\rangle = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu})\, \langle \underline{F}^q, \underline{w}\rangle, \tag{2.13}$$

where the coefficients $\Theta^q(\boldsymbol{\mu})$ and the $\boldsymbol{\mu}$-independent linear and bilinear forms are related to those appearing in (2.5). For example, $Q_a = \tilde{Q}_m + \tilde{Q}_n$, $\Theta_a^q(\boldsymbol{\mu}) = \tilde{\Theta}_m^q(\boldsymbol{\mu})$ and $\mathcal{A}^q(\underline{x}, \underline{w}) = m^q(y, z)$ for $1 \leq q \leq \tilde{Q}_m$, while $\Theta_a^{q+\tilde{Q}_m}(\boldsymbol{\mu}) = \tilde{\Theta}_n^q(\boldsymbol{\mu})$ and $\mathcal{A}^{q+\tilde{Q}_m}(\underline{x}, \underline{w}) = n^q(u, v)$ for $1 \leq q \leq \tilde{Q}_n$.

2.3. Truth approximation. Let $\mathcal{T}_{\mathcal{N}}$ be a triangulation of the domain Ω, we denote $V_{\mathcal{N}}^r$ the space of globally continuous functions that are polynomials of degree r on the single elements of the triangulation. Then we define $Y^{\mathcal{N}} = Y \cap V_{\mathcal{N}}^r$, $Q^{\mathcal{N}} \equiv Y^{\mathcal{N}}$ and $U_{\mathcal{N}} = U \cap V_{\mathcal{N}}^r$ in such a way that $X^{\mathcal{N}} = Y^{\mathcal{N}} \times U^{\mathcal{N}} \subset X$, $Q^{\mathcal{N}} \subset Q$ are sequences of FE approximation spaces. Moreover we indicate with $\mathcal{N}$ the global dimension – typically very "large" – of the product space $X^{\mathcal{N}} \times Q^{\mathcal{N}}$, i.e. $\mathcal{N} = \mathcal{N}_X + \mathcal{N}_Q$ where $\mathcal{N}_X = \mathcal{N}_Y + \mathcal{N}_U$ and $\mathcal{N}_Y = \mathcal{N}_Q$.

Following an *optimize-then-discretize* approach – rather than a *discretize-then-optimize* approach, see e.g. [10] – we introduce the *truth* Galerkin-FE approximation of the optimality system (2.9): given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu})) \in X^{\mathcal{N}} \times Q^{\mathcal{N}}$ such that

$$\begin{cases} \mathcal{A}(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w}\rangle & \forall \underline{w} \in X^{\mathcal{N}}, \\ \mathcal{B}(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in Q^{\mathcal{N}}. \end{cases} \tag{2.14}$$

Provided $Y^{\mathcal{N}} \equiv Q^{\mathcal{N}}$, the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ remains continuous over $X^{\mathcal{N}} \times X^{\mathcal{N}}$ and coercive over $X_0^{\mathcal{N}} = \{\underline{w} \in X^{\mathcal{N}} : \mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = 0 \quad \forall q \in Q^{\mathcal{N}}\}$, and the bilinear form $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ remains continuous and inf-sup stable over $X^{\mathcal{N}} \times Q^{\mathcal{N}}$, i.e. there exists a constant $\beta_0 > 0$ such that

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{q \in Q^{\mathcal{N}}} \sup_{\underline{w} \in X^{\mathcal{N}}} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{2.15}$$

In particular, mimicking the proof of Lemma 2.1 we can easily show that $\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$, being $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ the FE coercivity constant of the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu})$. Therefore, thanks to Brezzi theory, also the FE approximation (2.14) is well-posed.

Let us now investigate the structure of the algebraic system associated to the Galerkin approximation (2.14). We denote with $\{\underline{\varphi}_j \in X^{\mathcal{N}}\}_{j=1}^{\mathcal{N}_X}$, $\{\phi_k \in Q^{\mathcal{N}}\}_{k=1}^{\mathcal{N}_Q}$, the basis functions of the spaces $X^{\mathcal{N}}$, $Q^{\mathcal{N}}$, respectively. Then, (2.14) is equivalent to the linear system

$$\underbrace{\begin{pmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{\mathcal{K}(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \mathbf{p}^{\mathcal{N}}(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}(\boldsymbol{\mu}) \\ \mathbf{G}(\boldsymbol{\mu}) \end{pmatrix}, \tag{2.16}$$

where $\mathbf{x}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\mathbf{p}^{\mathcal{N}}(\boldsymbol{\mu})$ denotes the vectors of the coefficients in the expansion of $\underline{x}(\boldsymbol{\mu})$ and $p(\boldsymbol{\mu})$, while, for example, the elements of the matrix A are given by $A_{ij}(\boldsymbol{\mu}) = \mathcal{A}(\underline{\varphi}_j, \underline{\varphi}_i; \boldsymbol{\mu})$ for $1 \leq i,j \leq \mathcal{N}_X$. Let us notice that also the matrices appearing in (2.16) inherit the same affine decompositions (2.12), so that

$$A(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\, A^q, \qquad B(\boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\, B^q,$$

where the $\boldsymbol{\mu}$-independent matrices A^q, B^q represent the discrete counterparts of the corresponding bilinear. Analogously for the vectors $\mathbf{F}(\boldsymbol{\mu})$ and $\mathbf{G}(\boldsymbol{\mu})$.

For the resolution of the linear system (2.16) several strategies can be employed (see for instance [17, 1]): a popular alternative is based on the so called *reduced Hessian* methods, in which block elimination on the state and adjoint variables yields a reduced[5] system for the control variable whose matrix is the Schur complement of the optimality system. A radically alternative strategy consists of using *full space* (also called *all-at-once*) methods, where the optimality system is solved simultaneously for the state, adjoint and control variables. Both approaches present advantages and disadvantages and require problem-tailored design of suitable preconditioners and iterative linear solvers. Yet, beside the choice of the favorite solution algorithm, it is well known that the numerical solution of an optimal control problem entails large computational costs and may be very time-consuming already in the non-parametric case. Therefore, when performing the optimization process for many different parameter values or else when, for a new given configuration, the solution has to be computed in a rapid way, reducing the computational complexity is mandatory. This is why we advocate using suitable model order reduction techniques.

3. The reduced basis approximation. The idea of the RB method is to efficiently compute an approximation of $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu})$ by using approximation spaces made up of well-chosen solutions of (2.14), i.e. corresponding to specific choices of the parameter values. As already mentioned in the introduction, the main assumption is that the solution of (2.14) depends *smoothly* on the parameters, thus implying the parametric manifold $\mathcal{M}^{\mathcal{N}}$ to be smooth and approximable by selecting some *snapshot* FE solutions.

3.1. Construction of RB approximation spaces and stability properties. Let us suppose that we are given a set of hierarchical RB approximation subspaces $X_N \subset X^{\mathcal{N}}$ and $Q_N \subset Q^{\mathcal{N}}$, $N \in [1, N_{\max}]$, made up of properly selected FE solutions. By using Galerkin projection onto the low-dimensional subspace $X_N \times Q_N$, we obtain the following reduced basis approximation: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{A}(\underline{x}_N(\boldsymbol{\mu}), \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle & \forall \underline{w} \in X_N, \\ \mathcal{B}(\underline{x}_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q \rangle & \forall q \in Q_N. \end{cases} \tag{3.1}$$

[5]Here *reduced* must not be understood in the sense of *reduced order model.*

The existence, uniqueness and stability of the solution to problem (3.1) depend on the properties of the RB spaces X_N and Q_N, that are analyzed in the following.

Let us take, for given $N \in [1, N_{\max}]$, a finite set of parameter values $S_N = \{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N\}$ and consider the corresponding FE solutions $\{(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^n), p^{\mathcal{N}}(\boldsymbol{\mu}^n))\}_{n=1}^N$, the so called *snapshots* of the corresponding optimal control problem. We (naively) define the RB spaces for the state, control and adjoint variables respectively as

$$\begin{aligned} Y_N &= \mathrm{span}\{\zeta_n := y^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \\ U_N &= \mathrm{span}\{\lambda_n := u^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \\ Q_N &= \mathrm{span}\{\xi_n := p^{\mathcal{N}}(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \end{aligned} \tag{3.2}$$

and denote $X_N = Y_N \times U_N$. Let us discuss the well-posedness of the RB approximation (3.1). While the continuity properties of the bilinear forms over the RB spaces are automatically inherited from the parents spaces (i.e. the FE spaces), the coercivity property of the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ over

$$X_0^N = \{\underline{w} \in X_N \colon \mathcal{B}(\underline{w}, q; \boldsymbol{\mu}) = 0 \quad \forall q \in Q_N\}$$

and the fulfillment of the inf-sup condition of $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ are not granted and have to be proved. In particular, the problem (3.1) has to satisfy the following RB inf-sup condition: there exists $\beta_0 > 0$ such that

$$\beta_N(\boldsymbol{\mu}) = \inf_{q \in Q_N} \sup_{\underline{w} \in X_N} \frac{\mathcal{B}(\underline{w}, q; \boldsymbol{\mu})}{\|\underline{w}\|_X \|q\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{3.3}$$

The first idea in order to prove the fulfillment of (3.3) is to mimic the proof already used for the continuous problem and its FE approximation, see Lemma 2.1. Unfortunately, while in the continuous case (respectively for the FE approximation) the state and adjoint spaces Y and Q (respectively $Y^{\mathcal{N}}$ and $Q^{\mathcal{N}}$) are equivalent, with the choice (3.2) we lose this property on the corresponding RB spaces, i.e. $Y_N \neq Q_N$.

In order to recover the stability of the RB approximation, we therefore need to enrich in some way at least one of the RB spaces involved. This is not surprising when dealing with the RB approximation of a saddle-point problem, since the structure of this class of problems – in particular the requirement to fulfill the inf-sup condition – implies that building the RB approximation spaces solely from snapshots is not always sufficient. In fact, there are at least two other examples where a similar treatment shows to be necessary: the application of the RB method to parametrized Stokes equations [28, 32, 29, 8] and to parametrized variational inequalities [12]. Two possible strategies to achieve the stability of the approximation are either the use of a suitable supremizer operator or the use of the same (properly defined) space for the state and adjoint variables. While the first option can be seen as a trial to mimic what has been done in the case of the Stokes problem, the second option follows naturally from the discussion above and has been already considered in some previous works [5, 19] (even if not specifically for this reason). We chose to pursue the second one, being aware that these issues deserve further investigations in order to explore also other strategies, that might be more convenient from the computational point of view.

We thus define the following *aggregated* space for the state and adjoint variables

$$Z_N = \mathrm{span}\{\zeta_n := y^{\mathcal{N}}(\boldsymbol{\mu}^n), \, \xi_n := p^{\mathcal{N}}(\boldsymbol{\mu}), \quad n = 1, \dots, N\}, \tag{3.4}$$

and we let

$$Y_N = Z_N, \qquad X_N = Y_N \times U_N, \qquad Q_N = Z_N. \tag{3.5}$$

LEMMA 3.1. *If the reduced spaces X_N and Q_N are chosen as in (3.4)-(3.5), then the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ satisfies the inf-sup condition (3.3). Moreover we have the estimate*

$$\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}), \qquad \forall \boldsymbol{\mu} \in \mathcal{D},$$

where $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ is the coercivity constant associated to the FE approximation of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$.

Proof. It is sufficient to follow the proof of Lemma 2.1. In fact,

$$\beta_N(\boldsymbol{\mu}) = \inf_{q\in Q_N} \sup_{\underline{w}\in X_N} \frac{\mathcal{B}(\underline{w},q;\boldsymbol{\mu})}{\|\underline{w}\|_X\|q\|_Q} = \inf_{q\in Z_N} \sup_{(z,v)\in Z_N\times U_N} \frac{a(z,q;\boldsymbol{\mu}) - c(v,q;\boldsymbol{\mu})}{\|(z,v)\|_X\|q\|_Q}$$
$$\underset{(z,v)=(q,0)}{\geq} \inf_{q\in Z_N} \frac{a(q,q;\boldsymbol{\mu})}{\|q\|_Q} = \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) > 0.$$

Note that the choice $z = q$ is allowed because both z and q belong to the space Z_N. □

The well-posedness of the RB approximation is ensured by the following

PROPOSITION 3.2. *If the reduced spaces X_N and Q_N are chosen as in (3.4)-(3.5), then, for any $\boldsymbol{\mu} \in \mathcal{D}$, the RB approximation (3.1) has a unique solution $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ depending continuously on the data.*

Proof. It suffices to check that the assumptions of Brezzi theorem hold. As already mentioned, the continuity properties of the bilinear and linear forms over the RB space are automatically inherited from the parents spaces (i.e. the FE spaces). The fulfillment of the inf-sup condition of the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ has been proved in Lemma 3.1, while the fulfillment of the coercivity condition of the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ can be proved using the same arguments as in Lemma 2.1. □

3.2. Algebraic formulation and Offline-Online computational procedure. Let us now investigate the algebraic formulation associated to the enriched spaces introduced in the previous section. Let $\{\tau_j\}_{j=1}^{2N} = \{\zeta_j\}_{j=1}^{N} \cup \{\xi_j\}_{j=1}^{N}$ such that $Z_N = \text{span}\{\tau_j,\ j = 1,\dots,2N\}$, we can express the RB state, adjoint and control solutions as

$$\underline{x}_N(\boldsymbol{\mu}) = \sum_{j=1}^{3N} x_{Nj}(\boldsymbol{\mu})\underline{\sigma}_j, \qquad\qquad p_N(\boldsymbol{\mu}) = \sum_{j=1}^{2N} p_{Nj}(\boldsymbol{\mu})\tau_j.$$

where $\underline{\sigma}_j = (\tau_j, 0)$ for $j = 1,\dots,2N$, while $\underline{\sigma}_j = (0,\lambda_j)$ for $j = 2N+1,\dots,3N$, in such a way that $X_N = \text{span}\{\underline{\sigma}_j,\ j = 1,\dots,3N\}$. Hence, given a parameter $\boldsymbol{\mu}$, the RB solution of the problem (3.1) can be written as a combination of basis functions with weights given by the following reduced basis linear system:

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{\mathcal{K}_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}, \tag{3.6}$$

where $A_N(\boldsymbol{\mu}) = \sum \Theta_a^q(\boldsymbol{\mu})A_N^q$, $B_N(\boldsymbol{\mu}) = \sum \Theta_b^q(\boldsymbol{\mu})B_N^q$ and the submatrices A_N^q and B_N^q are given by $(A_N)_{ij}^q = \mathcal{A}^q(\underline{\sigma}_j,\underline{\sigma}_i)$, $(B_N)_{li}^q = \mathcal{B}^q(\underline{\sigma}_i,\tau_l)$, for $1 \leq i,j \leq 3N$, $1 \leq l \leq 2N$.

In order to state the connection between the RB linear system (3.6) and the FE discretization (2.16), let us define the *basis matrices* $Z_z = (\boldsymbol{\tau}_1 \mid \cdots \mid \boldsymbol{\tau}_N) \in \mathbb{R}^{\mathcal{N}\times 2N}$,

$Z_u = (\boldsymbol{\lambda}_1 \mid \cdots \mid \boldsymbol{\lambda}_N) \in \mathbb{R}^{\mathcal{N}\times N}$ and

$$Z_x = \begin{pmatrix} Z_z & 0 \\ 0 & Z_u \end{pmatrix} \in \mathbb{R}^{2\mathcal{N}\times 3N}, \qquad Z = \begin{pmatrix} Z_z & 0 & 0 \\ 0 & Z_u & 0 \\ 0 & 0 & Z_z \end{pmatrix} \in \mathbb{R}^{3\mathcal{N}\times 5N}.$$

Then, the matrix $\mathcal{K}_N = Z^T \mathcal{K} Z$ is given by

$$\mathcal{K}_N = \begin{pmatrix} A_N & B_N^T \\ B_N & 0 \end{pmatrix} = \begin{pmatrix} Z_x^T A Z_x & Z_x^T B^T Z_z \\ Z_z^T B Z_x & 0 \end{pmatrix}. \tag{3.7}$$

Thus the matrix $\mathcal{K}_N$ is still symmetric, with saddle-point structure and has dimension $5N \times 5N$. Although being dense (rather than sparse as in the FE case), the system matrix is very small, with a size independent of the FE space dimension $\mathcal{N}$; for this reason the RB linear system can be easily solved using direct solvers. Furthermore, to keep under control the condition number of the matrix $\mathcal{K}_N$ we have adopted the Gram-Schmidt orthonormalization procedure [30]. In particular we apply the Gram-Schmidt procedure separately on the basis functions of the space Z_N and on the basis functions of the space U_N.

Thanks to the assumption of affine parameter dependence, we can decouple the formation of the matrix $\mathcal{K}_N(\boldsymbol{\mu})$ in two stages, the Offline and Online stages, that enable the efficient resolution of the system (3.6) for each new parameter $\boldsymbol{\mu}$.
In particular, in the *Offline stage*, performed only once, we first compute and store the basis function $\{\tau_i\}_{i=1}^{2N}$ and $\{\lambda_j\}_{j=1}^{N}$, and form the $\boldsymbol{\mu}$-independent matrices A_N^q, $1 \le q \le Q_a$, B_N^q, $1 \le q \le Q_b$ and the vectors F_N^q, $1 \le q \le Q_f$, G_N^q, $1 \le q \le Q_g$. The operation count depends on N, Q_a, Q_b, Q_f, Q_g and $\mathcal{N}$.
In the *Online stage*, performed for each new value $\boldsymbol{\mu}$, we use the precomputed matrices A_N^q, B_N^q and vectors F_N^q, G_N^q to assemble the (full) matrix $\mathcal{K}_N$ and the vectors $\mathbf{F}_N$, $\mathbf{G}_N$ appearing in (3.6); we then solve the resulting system to obtain $(\mathbf{x}_N, \mathbf{p}_N)$. The Online operation count depends on N, Q_a, Q_b, Q_f, Q_g but is independent of $\mathcal{N}$. In particular we need $O((Q_a+Q_b)N^2)$ and $O((Q_f+Q_g)N)$ operations to assemble matrices and vectors, and $O((5N)^3)$ operations to solve the RB linear system (3.6).

3.3. Sampling strategy. For the construction of the hierarchical Lagrange RB approximation spaces – and thus the optimal choice of the sample points $\boldsymbol{\mu}^n$, $1 \le n \le N$ – we rely on the sampling strategy based on the standard *greedy algorithm* [30, 29]. Let $\Xi_{\text{train}} \subset \mathcal{D}$ be a finite dimensional sample set, called the set of *train* samples. The cardinality of Ξ_{train} will be denoted with n_{train}, that we assume to be sufficiently large such that Ξ_{train} be a good approximation of the set $\mathcal{D}$ (a finite dimensional surrogate for $\mathcal{D}$). The idea of the greedy procedure is that, starting with a train sample Ξ_{train}, we adaptively select (in the sense of minimizing a suitable error indicator) N parameters $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N$ and form the hierarchical sequence of reduced basis spaces X_N, Q_N as in (3.4)-(3.5). At each iteration N, the greedy algorithm appends to the previously *retained* snapshots that particular candidate – over all candidate snapshots $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}))$, $\boldsymbol{\mu} \in \Xi_{\text{train}}$ – which is least well approximated by the "old" RB space $X_{N-1} \times Q_{N-1}$. The key ingredient of this adaptive procedure is a rigorous, sharp and inexpensive estimator $\Delta_N(\boldsymbol{\mu})$ for the RB error such that

$$\left(\|\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \underline{x}_N(\boldsymbol{\mu})\|_X^2 + \|p^{\mathcal{N}}(\boldsymbol{\mu}) - p_N(\boldsymbol{\mu})\|_Q^2\right)^{1/2} \le \Delta_N(\boldsymbol{\mu}), \tag{3.8}$$

where $(\underline{x}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))$ is the RB approximated solution associated with the generic RB space $X_N \times Q_N$. The construction of the a posteriori error estimator Δ_N will be described in detail in §4.

Given such an estimator, we can state precisely the steps required by the greedy algorithm. By denoting ε_{tol} a chosen tolerance for the stopping criterium, the greedy sampling strategy can be implemented as reported in Algorithm 1.

$S_1 = \{\boldsymbol{\mu}^1\}$, compute $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^1), p^{\mathcal{N}}(\boldsymbol{\mu}^1))$ by solving the *truth* approximation (2.14)
$U_1 = \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^1)\}, \quad Z_1 = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^1), p^{\mathcal{N}}(\boldsymbol{\mu}^1)\}$
$X_1 = Z_1 \times U_1, \quad Q_1 = Z_1$
for $N = 2 : N_{\max}$ **do**
 $\boldsymbol{\mu}^N = \arg\max_{\boldsymbol{\mu}\in\Xi_{\text{train}}} \Delta_{N-1}(\boldsymbol{\mu})$
 $\varepsilon_{N-1} = \Delta_{N-1}(\boldsymbol{\mu})$
 if $\varepsilon_{N-1} \leq \varepsilon_{\text{tol}}$
  $N_{\max} = N - 1$
 end if
 compute $(\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}^N), p^{\mathcal{N}}(\boldsymbol{\mu}^N))$ by solving the *truth* approximation (2.14)
 $S_N = S_{N-1} \cup \{\boldsymbol{\mu}^N\}$
 $U_N = U_{N-1} \cup \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^N)\}, \quad Z_N = Z_{N-1} \cup \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^N), p^{\mathcal{N}}(\boldsymbol{\mu}^N)\}$
 $X_N = Z_N \times U_N, \quad Q_N = Z_N$
end for

Algorithm 1: Greedy algorithm for parametrized optimal control problems.

We underline again that the key point in the algorithm is to exploit an a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ efficiently computable, since at each iteration the algorithm requires to evaluate $\Delta_N(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \Xi_{\text{train}}$.

4. Rigorous *a posteriori* error estimates. In the RB framework a posteriori error estimates plays a crucial role in order to guarantee the efficiency and reliability of the method. As regards efficiency, the error bound is essential in the sampling procedure, by allowing an exhaustive exploration of the parameters domain and a proper selection of the basis functions. As regards reliability, at the Online stage for each new value of parameter $\boldsymbol{\mu} \in \mathcal{D}$, the a posteriori estimator permits to bound the error of the RB approximation with respect to the underlying truth approximation.

Different strategies can be pursued in order to provide a posteriori error estimation for parametrized optimal control problems. In [5] an efficient yet not rigorous estimator has been proposed dealing with time-dependent optimal control problems, while recently in [19] similar techniques combined with some previous results proposed in [36] have been applied to the same problem considered here, providing an efficient and rigorous estimator. In this work, we propose a new a posteriori error estimate that can be easily obtained exploiting the structure of the optimality system. In particular, once the saddle-point structure of the optimality system has been highlighted, one can apply three different approaches, already proposed in the RB context: (i) to exploit Brezzi stability theory [4]; (ii) to use the Nečas-Babuška stability theory [2, 23]; (iii) or to adopt a penalty approach [11]. While the approaches (i) and (iii) have been only recently applied in the RB context, respectively in [8] and [7], the second approach is quite standard in the RB context [30]. We thus choose to pursue the latter, exploiting the analogies with the RB scheme proposed for affinely parametrized Stokes equations in [32, 29].

In §4.1 we construct a rigorous and inexpensive (i.e. $\mathcal{N}$-independent) a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ such that

$$\left(\|\underline{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \underline{x}_N(\boldsymbol{\mu})\|_X^2 + \|p^{\mathcal{N}}(\boldsymbol{\mu}) - p_N(\boldsymbol{\mu})\|_Q^2\right)^{1/2} \leq \Delta_N(\boldsymbol{\mu}). \tag{4.1}$$

Then in §4.2, using the same ingredients, we construct a rigorous and inexpensive a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ for the error on the cost functional, i.e.

$$|J(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) - J(y_N(\boldsymbol{\mu}), u_N(\boldsymbol{\mu}); \boldsymbol{\mu})| \leq \Delta_N^J(\boldsymbol{\mu}). \tag{4.2}$$

4.1. Bound for the solution. Since saddle point problems can be regarded as a particular case of *weakly coercive* (also called *noncoercive*) problems, the construction of the error estimator $\Delta_N(\boldsymbol{\mu})$ can be carried out by using the Nečas-Babuška stability theory [2, 23].

Upon defining the space $\mathcal{X} = X \times Q$, the bilinear form $\mathsf{B}(\cdot, \cdot; \boldsymbol{\mu})\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$,

$$\mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu}) := \mathcal{A}(\underline{x}, \underline{w}; \boldsymbol{\mu}) + \mathcal{B}(\underline{w}, p; \boldsymbol{\mu}) + \mathcal{B}(\underline{x}, q; \boldsymbol{\mu}), \tag{4.3}$$

and the linear continuous functional $\mathsf{F}(\cdot; \boldsymbol{\mu})\colon \mathcal{X} \to \mathbb{R}$,

$$\mathsf{F}(\mathsf{w}; \boldsymbol{\mu}) = \langle \underline{F}(\boldsymbol{\mu}), \underline{w} \rangle + \langle G(\boldsymbol{\mu}), q \rangle, \tag{4.4}$$

where $\mathsf{x} = (\underline{x}, p) \in \mathcal{X}$ and $\mathsf{w} = (\underline{w}, q) \in \mathcal{X}$, problem (2.9) can equivalently be reformulated as: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\text{find } \mathsf{x} \in \mathcal{X} \text{ s.t:} \qquad \mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu}) = \mathsf{F}(\mathsf{w}; \boldsymbol{\mu}) \qquad \forall \mathsf{w} \in \mathcal{X}. \tag{4.5}$$

According to Nečas theorem, the problem (4.5) is well posed if for any $\boldsymbol{\mu} \in \mathcal{D}$ the bilinear form $\mathsf{B}(\cdot, \cdot; \boldsymbol{\mu})$ is continuous and weakly coercive, i.e. there exists a constant $\hat{\beta}_0 > 0$ such that[6]

$$\hat{\beta}(\boldsymbol{\mu}) = \inf_{\mathsf{w} \in \mathcal{X}} \sup_{\mathsf{x} \in \mathcal{X}} \frac{\mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}} \|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0. \tag{4.6}$$

Moreover, holding these assumptions, for any $\boldsymbol{\mu} \in \mathcal{D}$ the unique solution satisfies the following stability estimate

$$\|\mathsf{x}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{1}{\hat{\beta}(\boldsymbol{\mu})} \|\mathsf{F}(\cdot; \boldsymbol{\mu})\|_{\mathcal{X}'}. \tag{4.7}$$

Actually, since the bilinear forms $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ and $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ satisfy the hypotheses of Brezzi theorem, it can be shown (see e.g. [37, 11]) that the the compound form $\mathsf{B}(\cdot, \cdot; \boldsymbol{\mu})$ is bounded and weakly coercive. Similarly, the FE and RB approximations satisfy the same inf-sup condition,

$$\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) := \inf_{\mathsf{w} \in \mathcal{X}^{\mathcal{N}}} \sup_{\mathsf{x} \in \mathcal{X}^{\mathcal{N}}} \frac{\mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}} \|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0^{\mathcal{N}} > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{4.8}$$

$$\hat{\beta}_N(\boldsymbol{\mu}) := \inf_{\mathsf{w} \in \mathcal{X}_N} \sup_{\mathsf{x} \in \mathcal{X}_N} \frac{\mathsf{B}(\mathsf{x}, \mathsf{w}; \boldsymbol{\mu})}{\|\mathsf{x}\|_{\mathcal{X}} \|\mathsf{w}\|_{\mathcal{X}}} \geq \hat{\beta}_0^N > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{4.9}$$

where $\mathcal{X}^{\mathcal{N}} = X^{\mathcal{N}} \times Q^{\mathcal{N}}$ and $\mathcal{X}_N = X_N \times Q_N$. Moreover the stability estimate (4.7) holds also for the FE and RB approximations, in particular

$$\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \frac{1}{\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})} \|\mathsf{F}(\cdot; \boldsymbol{\mu})\|_{\mathcal{X}'}, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{4.10}$$

[6] In the following we will refer to the inf-sup constant $\hat{\beta}(\boldsymbol{\mu})$ (4.6) as the Babuška inf-sup constant, in contrast to the Brezzi inf-sup constant $\beta(\boldsymbol{\mu})$ (2.8); similar notation will be used for their FE and RB approximations.

The construction of the a posteriori error estimation is based on two main ingredients (as usual in RB context): an effective calculation of a lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ and the calculation of the dual norm of the residual [24]. As regards the first one, we suppose to have at our disposal a $\boldsymbol{\mu}$-dependent lower bound $\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) : \mathcal{D} \to \mathbb{R}$ such that

$$\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) \geq \hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) \geq \hat{\beta}_0 > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{4.11}$$

and the Online computational time to evaluate $\boldsymbol{\mu} \to \hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})$ is independent of $\mathcal{N}$. The calculation of $\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu})$ can be carried out using the Natural Norm Successive Constraint Method, an improvement of the SCM algorithm specifically tailored for noncoercive problems, see e.g. [16, 29] for a detailed explanation of this procedure as well as for many numerical tests.
As regards the second ingredient, the residual $\mathsf{r}(\cdot;\boldsymbol{\mu}) \in (\mathcal{X}^{\mathcal{N}})'$ is defined as

$$\mathsf{r}(\mathsf{w};\boldsymbol{\mu}) := \mathsf{F}(\mathsf{w};\boldsymbol{\mu}) - \mathsf{B}(\mathsf{x}_N, \mathsf{w};\boldsymbol{\mu}) \qquad \forall \mathsf{w} \in \mathcal{X}^{\mathcal{N}}.$$

Finally, let us define the error between the "truth" FE approximation and the RB approximation, $\mathsf{e}(\boldsymbol{\mu}) := \mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})$. We can now formulate an a posteriori estimator for the error $\mathsf{e}(\boldsymbol{\mu})$.

PROPOSITION 4.1. *For any given $\boldsymbol{\mu} \in \mathcal{D}$, $N \in [1, N_{max}]$, and $\hat{\beta}_{LB}(\boldsymbol{\mu})$ satisfying (4.11), we define*

$$\Delta_N(\boldsymbol{\mu}) = \frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}}{\hat{\beta}_{LB}(\boldsymbol{\mu})}. \tag{4.12}$$

Then, $\Delta_N(\boldsymbol{\mu})$ is an upper bound for the error $\mathsf{e}(\boldsymbol{\mu})$,

$$\|\mathsf{e}(\boldsymbol{\mu})\|_{\mathcal{X}} \leq \Delta_N(\boldsymbol{\mu}), \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \quad \forall N \in [1, N_{max}]. \tag{4.13}$$

Proof. The problem statement for the FE solution $\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu})$ and for the RB solution $\mathsf{x}_N(\boldsymbol{\mu})$ and the bilinearity of $\mathsf{B}(\cdot,\cdot;\boldsymbol{\mu})$ imply that the error $\mathsf{e}(\boldsymbol{\mu})$ statisfy the following equation: $\mathsf{B}(\mathsf{e}(\boldsymbol{\mu}), \mathsf{w};\boldsymbol{\mu}) = \mathsf{r}(\mathsf{w};\boldsymbol{\mu})$, $\forall \mathsf{w} \in \mathcal{X}^{\mathcal{N}}$. Then it suffices to apply the stability estimate (4.7) and exploit the lower bound (4.11) for the Babuška inf-sup constant. ☐

As usual (see for instance [30, 29]), the computation of the dual norm of the residual can be decomposed in two stages: an expensive, $\boldsymbol{\mu}$-independent Offline stage and an inexpensive Online stage. As a result, given $\boldsymbol{\mu} \in \mathcal{D}$, the evaluation of $\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}$ requires $O(25N^2Q_B^2 + 5NQ_BQ_f + Q_F^2)$ operations, independent of $\mathcal{N}$.

4.2. A posteriori error bound for the cost functional. To develop an a posteriori error bound on the cost functional $J(y, u;\boldsymbol{\mu})$, we firstly observe that this is equivalent to provide an estimator for the error on $\mathcal{J}(\underline{x};\boldsymbol{\mu})$, since $\mathcal{J}(\cdot;\boldsymbol{\mu})$ and $J(\cdot,\cdot;\boldsymbol{\mu})$ differ only in a constant term once $\boldsymbol{\mu} \in \mathcal{D}$ is fixed. Although the cost functional $\mathcal{J}(\cdot;\boldsymbol{\mu})$ is a *quadratic* functional, thanks to the structure of the optimal control problem we can avoid to use the techniques of error estimation for quadratic outputs already proposed in the RB context, see for instance [34, 14, 22]. Rather, following the work in [5] we may use a goal-oriented analysis, a standard tool for the development of a posteriori error estimates for optimal control problems.

The error on the cost functional evaluated with respect to the FE and RB approximations will be denoted with

$$\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu}) = J(y^{\mathcal{N}}(\boldsymbol{\mu}), u^{\mathcal{N}}(\boldsymbol{\mu});\boldsymbol{\mu}) - J(y_N(\boldsymbol{\mu}), u_N(\boldsymbol{\mu});\boldsymbol{\mu}).$$

Recalling the definition of the Lagrangian functional (2.10), we observe that we can use a different formalism to express the gradient of the Lagrangian as

$$\nabla\mathcal{L}(\mathsf{x};\boldsymbol{\mu})[\mathsf{w}] = \mathsf{B}(\mathsf{x},\mathsf{w};\boldsymbol{\mu}) - \mathsf{F}(\mathsf{w};\boldsymbol{\mu}), \qquad \forall \mathsf{w} \in \mathcal{X}. \tag{4.14}$$

Then, we can show the following result.

PROPOSITION 4.2. *For any given $\boldsymbol{\mu} \in \mathcal{D}$, $N \in [1, N_{max}]$, and $\hat{\beta}_{LB}(\boldsymbol{\mu})$ satisfying (4.11), we define*

$$\Delta_N^J(\boldsymbol{\mu}) = \frac{1}{2}\frac{\|\mathsf{r}(\cdot;\boldsymbol{\mu})\|^2_{\mathcal{X}'}}{\hat{\beta}_{LB}(\boldsymbol{\mu})}. \tag{4.15}$$

Then, $\Delta_N^J(\boldsymbol{\mu})$ is an upper bound for the error on the cost functional,

$$|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})| \le \Delta_N^J(\boldsymbol{\mu}), \qquad \forall \boldsymbol{\mu} \in \mathcal{D}, \quad \forall N \in [1, N_{max}]. \tag{4.16}$$

Proof. The RB error on the cost functional can be rewritten as (see e.g. [3, 5])

$$\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu}) = \frac{1}{2}\nabla\mathcal{L}(\mathsf{x}_N(\boldsymbol{\mu});\boldsymbol{\mu})[\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})].$$

Thanks to (4.14) we have that

$$\nabla\mathcal{L}(\mathsf{x}_N;\boldsymbol{\mu})[\mathsf{x}^{\mathcal{N}} - \mathsf{x}_N] = \mathsf{B}(\mathsf{x}_N, \mathsf{x}^{\mathcal{N}} - \mathsf{x}_N;\boldsymbol{\mu}) - \mathsf{F}(\mathsf{x}^{\mathcal{N}} - \mathsf{x}_N;\boldsymbol{\mu}) = \mathsf{r}(\mathsf{x}^{\mathcal{N}} - \mathsf{x}_N;\boldsymbol{\mu}).$$

By exploiting the continuity of the residual $\mathsf{r}(\cdot;\boldsymbol{\mu})$ and the estimate (4.13) we obtain the required bound (4.16). □

Note that the error estimator $\Delta_N^J(\boldsymbol{\mu})$ does not need any further ingredients besides those already available: the efficient computation of the dual norm of the residual and the calculation of a lower bound for the Babuška inf-sup constant.

5. Numerical examples. In this section we discuss three numerical examples in order to verify the properties – and to test the performances – of the proposed RB scheme. In the cases in which we consider a parametrized geometry we firstly define an "original" problem (subscript o) posed over a parameters dependent domain, then we trace back the problem to a reference domain through suitable affine geometrical mappings (see [30, 29, 22] for the details) in order to recover the formulation (2.9). The implementation of the method has been carried out in the MATLAB® environment using an enhanced version of the `rbMIT` library [15][7].

5.1. Test 1: distributed optimal control for the Laplace equation with geometrical parametrization. We consider an "original" domain $\Omega_o(\boldsymbol{\mu}) = \Omega_o^1 \cup \Omega_o^2(\boldsymbol{\mu})$ given by a rectangle separated in two subdomains, with the first one parameter independent, as shown in Figure 5.1. We consider two parameters $\boldsymbol{\mu} = (\mu_1, \mu_2)$, being μ_1 related to the geometry of Ω_o^2 while μ_2 is such that $y_d(\boldsymbol{\mu}) = 1$ in Ω_o^1 and $y_d(\boldsymbol{\mu}) = \mu_2$ in $\Omega_o^2(\boldsymbol{\mu})$, i.e. the observation function is parameter dependent (constant on each subdomain). The set spanned by the parameters is given by $\mathcal{D} = [1, 3.5] \times [0.5, 2.5]$.

[7]Since the problems we deal with are of small size, all the required linear systems (in particular in the Offline stage) will be solved using the direct solver provided by MATLAB. All the computations are performed on a personal computer with an Intel Core i5-2400S CPU and 16 GB of RAM.

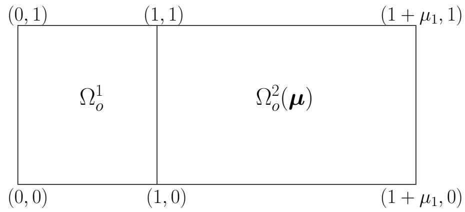

FIG. 5.1. *Test 1: "original" domain* $\Omega_o(\boldsymbol{\mu})$.

We consider the following optimal control problem:

$$
\begin{aligned}
&\min_{y_o, u_o} J(y_o(\boldsymbol{\mu}), u_o(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\|y_o(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\Omega_o)} + \frac{\alpha}{2}\|u_o(\boldsymbol{\mu})\|^2_{U_o}, \\
&\text{s.t.} \begin{cases} -\Delta y_o(\boldsymbol{\mu}) = u_o(\boldsymbol{\mu}) & \text{in } \Omega_o(\boldsymbol{\mu}), \\ y_o(\boldsymbol{\mu}) = g_D & \text{on } \Gamma^o_D(\boldsymbol{\mu}) = \partial\Omega_o(\boldsymbol{\mu}), \end{cases}
\end{aligned} \tag{5.1}
$$

where y_o and u_o are the state and control functions defined on the original domain, while the Dirichlet boundary condition is given by $g_D = 1$. We denote with Y_o and U_o the spaces $H^1_0(\Omega_o)$ and $L^2(\Omega_o)$ respectively, moreover $Q_o \equiv Y_o$. By tracing the problem back to a reference domain $\Omega = \Omega_o(\boldsymbol{\mu}_{\text{ref}})$ (with the arbitrary choice $\boldsymbol{\mu}_{\text{ref}} = (1, 1)$) we obtain the parametrized formulation (2.9) where the affine decompositions (2.12) (2.13) hold with $Q_a = 2$, $Q_b = 3$, $Q_f = 2$, $Q_g = 3$.

Computations are based upon a finite element approximation on $\mathbb{P}^1$ spaces for the state, control and adjoint variables; the total number of degrees of freedom, i.e. the dimension of the space $\mathcal{X}^{\mathcal{N}} = Y^{\mathcal{N}} \times U^{\mathcal{N}} \times Q^{\mathcal{N}}$, is $\mathcal{N} = 5\,982$, obtained using a mesh of 4 136 triangular elements. The regularization parameter is kept fixed and equal to $\alpha = 0.01$. In Figure 5.2 a representative solution for a fixed value of the parameters is given.

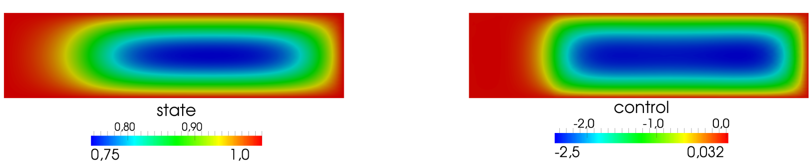

FIG. 5.2. *Test 1: representative solution for* $\boldsymbol{\mu} = (0.6, 3)$*; on the left the state variable* y_N*, on the right the optimal control* u_N*.*

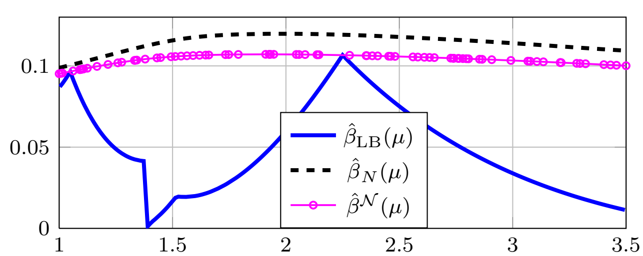

FIG. 5.3. *Test 1: lower bound for the Babuška inf-sup constant* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of the geometrical parameter* μ_1 *(on the x-axis).*

With a fixed tolerance $\varepsilon_{tol} = 5 \cdot 10^{-4}$, $N_{max} = 12$ basis functions have been selected by the greedy algorithm, thus resulting in a RB linear system of dimension 60×60. In Figure 5.3 we show the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ (defined in (4.8)) obtained using the natural norm SCM algorithm, which requires in this

case the solution of $10 + 2Q_B$ eigenproblems of dimension $\mathcal{N}$ (see [16, 29] for further details). In Figure 5.3 the RB Babuška inf-sup constant $\hat{\beta}_N(\boldsymbol{\mu})$ defined in (4.9) is also reported, in particular we can observe that $\hat{\beta}_N(\boldsymbol{\mu}) \geq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, thus indicating the good stability property of the RB approximation.
Furthermore, as regards the stability properties, in Figure 5.4 we give some numerical results on the discrete Brezzi inf-sup constants $\beta^{\mathcal{N}}(\boldsymbol{\mu})$ and $\beta_N(\boldsymbol{\mu})$, also compared with the coercivity constant $\tilde{\alpha}(\boldsymbol{\mu})$ of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$ in the state equation. In Figure 5.4a we report some results obtained in a preliminary numerical investigation without any enrichment option, i.e. using different RB spaces Y_N and Q_N (see §3.1). We compare the discrete Brezzi inf-sup constant and coercivity constant for the FE and RB approximation. We can confirm that, as claimed in §2.3 (see also Lemma 2.1), $\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$. Moreover we observe that

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) \geq \beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}),$$

hence (as expected) we cannot bound from below the RB inf-sup constant $\beta_N(\boldsymbol{\mu})$ with similar quantities related to the FE approximations. We note also that in this case the RB coercivity constant $\tilde{\alpha}_N(\boldsymbol{\mu})$ is in fact an inf-sup constant, since we are approximating the state equation with a Petrov-Galerkin scheme, i.e.

$$\tilde{\alpha}_N(\boldsymbol{\mu}) = \inf_{q \in Q_N} \sup_{y \in Y_N} \frac{a(y,q;\boldsymbol{\mu})}{\|q\|_Q \|y\|_Y}, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

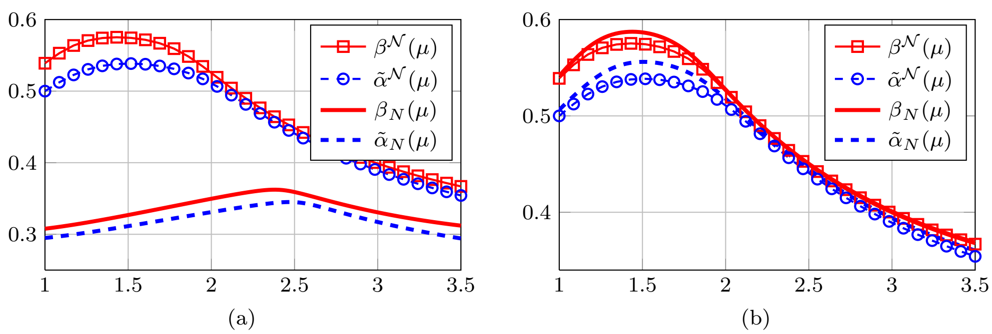

FIG. 5.4. *Test 1: comparison of the FE and RB discrete Brezzi inf-sup constant $\beta(\boldsymbol{\mu})$ and coercivity constant of the state equation $\tilde{\alpha}(\boldsymbol{\mu})$. The two quantities are given as function only of μ_1, since μ_2 does not appear in the affine expansion of $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$. (a) No enrichment: $Y_N \neq Q_N$. (b) Aggregated space: $Y_N = Q_N = Z_N$ with Z_N defined as in* (3.4).

In Figure 5.4b we compare the RB stability factors obtained using the aggregated space Z_N for the state and adjoint variables. In this case we have a numerical evidence of the result proven in Lemma 3.1, that is

$$\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

Finally in Figure 5.5 we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$ and the a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ with the true error on the cost functional $|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})|$.

As regards the computational performances, the Offline computational time is equal to $t_{RB}^{offline} = 139s$, the (average) Online evaluation time is $t_{RB}^{online} = 8.5$ ms comprehensive of the evaluation of the a posteriori error estimation; we remark that most of the Offline time is spent performing the SCM and greedy algorithms, the former requiring around 88 seconds while the latter requiring around 46 seconds. The evaluation time for the FE approximation is equal to about $t_{FE}^{online} = 1$ s taking into account the time needed for assembling the FE matrices and vectors.

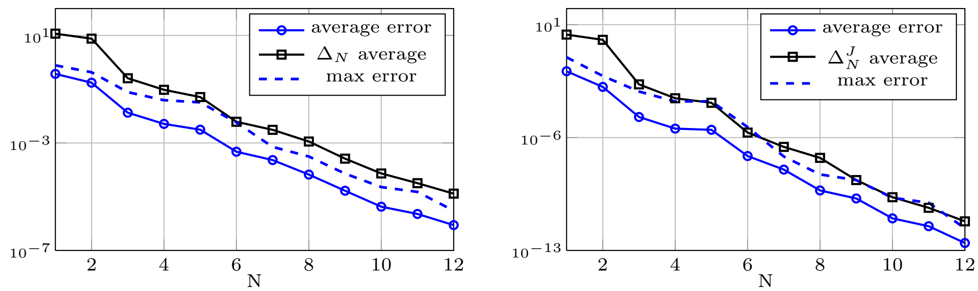

FIG. 5.5. *Test 1. Average and max computed errors and estimate between the* truth *FE solution and the RB approximation, for $N = 1, \cdots, N_{max}$ (left). Average computed errors and estimate $\Delta_N^J(\mu)$ between $\mathcal{J}^{\mathcal{N}}(\mu)$ and $\mathcal{J}_N(\mu)$, for $N = 1, \cdots, N_{max}$ (right). Here Ξ_{train} is a sample of size $n_{train} = 1000$ and $N_{max} = 12$.*

5.2. Test 2: distributed optimal control for a Graetz convection-diffusion problem with physical parametrization. As a second example we consider a distributed optimal control problem for the Graetz conduction-convection equation. With respect to the previous test we consider here a simple physical parametriza-

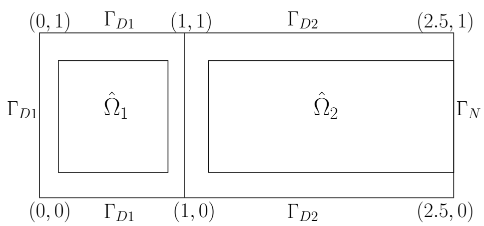

FIG. 5.6. *Test 2: domain Ω (the observations subdomains are denoted with $\hat{\Omega}_1$ and $\hat{\Omega}_2$).*

tion instead of a geometrical one; in particular, μ_1 will be the Péclet number, while μ_2 and μ_3, similarly to the previous example, are such that $y_d(\boldsymbol{\mu}) = \mu_2$ in $\hat{\Omega}_1$ and $y_d(\boldsymbol{\mu}) = \mu_3$ in $\hat{\Omega}_2$, where the spatial domain (shown in Figure 5.6) is the rectangle $\Omega = [0, 2.5] \times [0, 1]$. The parameter domain is $\mathcal{D} = [3, 20] \times [0.5, 1.5] \times [1.5, 2.5]$. We consider the following optimal control problem:

$$\min_{y,u} J(y, u; \boldsymbol{\mu}) = \frac{1}{2}\|y(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega})} + \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{L^2(\Omega)},$$

$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_1}\Delta y(\boldsymbol{\mu}) + x_2(1 - x_2)\dfrac{\partial y(\boldsymbol{\mu})}{\partial x_1} = u(\boldsymbol{\mu}) & \text{in } \Omega \\ \dfrac{1}{\mu_1}\nabla y(\mu) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_N \\ y(\boldsymbol{\mu}) = 1 \quad \text{on } \Gamma_{D1}, \qquad y(\boldsymbol{\mu}) = 2 & \text{on } \Gamma_{D2}, \end{cases} \tag{5.2}$$

where $y(\boldsymbol{\mu})$ is the temperature field, the control $u(\boldsymbol{\mu})$ acts as a heat source and $\hat{\Omega} = \hat{\Omega}_1 \cup \hat{\Omega}_2$ is the observation domain. The problem admits an affine decomposition with $Q_a = 1$, $Q_b = 2$, $Q_f = 2$, $Q_g = 2$ components. For the computation we fixed $\alpha = 0.01$ and used piecewise linear finite elements for the FE approximation, the dimension of the global FE space $\mathcal{X}^{\mathcal{N}}$ used is $\mathcal{N} = 10\,494$.

TABLE 5.1
Numerical details for Test 2. The RB spaces have been built by means of the greedy procedure and $N = 19$ basis functions have been selected.

Approximation data		Computational performances	
Number of FE dof $\mathcal{N}$	10 494	Linear system size reduction	110:1
Number of parameters P	3	Offline total time	417 s
Error tolerance greedy ε_{tol}	10^{-4}	Offline SCM time	315 s
Affine operator components Q_B	3	Offline greedy time	90 s

With a fixed tolerance $\varepsilon_{tol}^{rel} = 10^{-4}$, $N_{max} = 19$ basis functions have been selected, thus resulting in a RB linear system of dimension 95×95. In Figure 5.7a we show the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ obtained using the natural norm SCM algorithm; SCM requires in this case the solution of $28 + 2Q_B$ eigenproblems. Once again we can observe that $\hat{\beta}_N(\boldsymbol{\mu}) \geq \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, thus indicating the good stability property of the RB approximation.

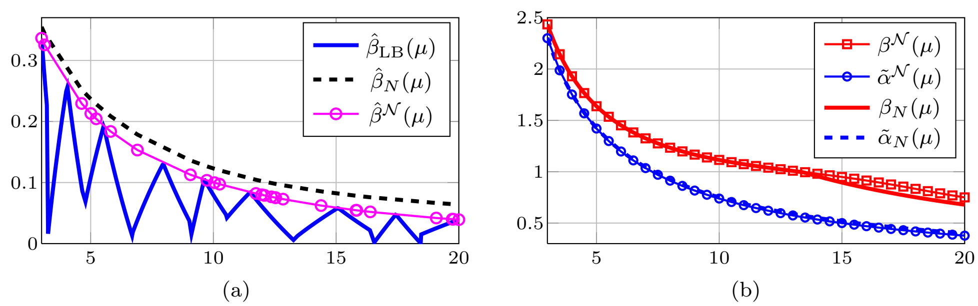

FIG. 5.7. *Test 2: stability factors as functions of the physical parameter μ_1. (a) Lower bound for the discrete Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. (b) Comparison of discrete Brezzi inf-sup constant $\beta(\boldsymbol{\mu})$ and coercivity constant $\tilde{\alpha}(\boldsymbol{\mu})$ for the FE and RB approximations.*

In Figure 5.7b we compare the Brezzi inf-sup constants $\beta^{\mathcal{N}}(\boldsymbol{\mu})$ and $\beta_N(\boldsymbol{\mu})$ and the coercivity constants $\tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\tilde{\alpha}_N(\boldsymbol{\mu})$ of the bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$. As in the previous example we have confirmed numerically that $\beta_N(\boldsymbol{\mu}) \geq \tilde{\alpha}_N(\boldsymbol{\mu}) \geq \tilde{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$. Finally in Figure 5.8 we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$ and the a posteriori error bound $\Delta_N^J(\boldsymbol{\mu})$ with the true error on the cost functional $|\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})|$.

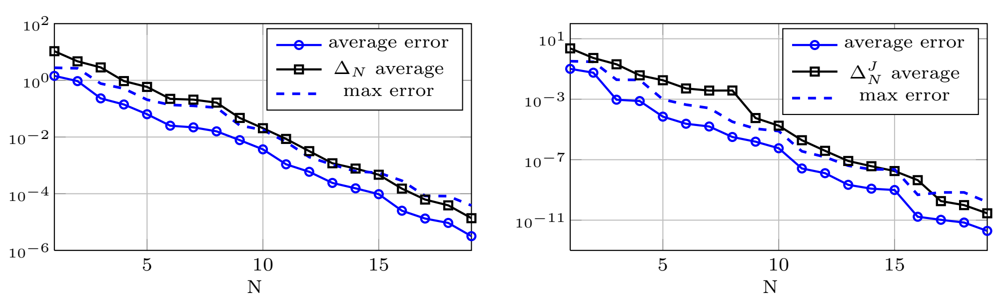

FIG. 5.8. *Test 2. Average and max computed errors and bound between the* truth *FE solution and the RB approximation (left). Average true error and bound $\Delta_N^J(\mu)$ between $\mathcal{J}^{\mathcal{N}}(\mu)$ and $\mathcal{J}_N(\mu)$ (right).*

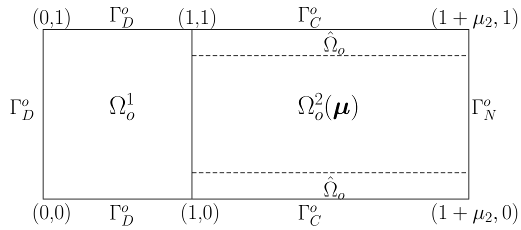

FIG. 5.9. *Test 3: "original" domain $\Omega_o(\boldsymbol{\mu})$.*

As regards the computational performances, while the average Online time needed to compute and certify the RB solution is approximately equal to the one reported in the previous test, the Offline computational time required to build all the ingredients is now equal to $t_{RB}^{offline} = 417s$. Notice that here performing the SCM algorithm requires around the 75% of the overall Offline time, a percentage that can further increase rapidly when the number of parameters P, the number of terms Q_B in the affine decomposition or the number of FE degrees of freedom $\mathcal{N}$ increase. In the next example we will discuss an alternative strategy for the construction of the lower bound $\hat{\beta}_{LB}(\boldsymbol{\mu})$, in order to avoid this computational bottleneck in the Offline stage.

5.3. Test 3: boundary optimal control for a Graetz flow with both physical and geometrical parametrization. This third example deals again with a control problem for a Graetz flow, however this time we consider a boundary control instead of a distributed one and we consider both a geometrical and physical parametrization. The original domain is shown in Figure 5.9, we consider 3 parameters: μ_1 is the Péclet number, μ_2 is the geometrical parameter (the length of second portion of the channel) and μ_3 is such that $y_d(\boldsymbol{\mu}) = \mu_3 \chi_{\hat{\Omega}_o}$, being $\hat{\Omega}_o(\boldsymbol{\mu})$ the observation domain $\hat{\Omega}_o(\boldsymbol{\mu}) \subset \Omega_o^2(\boldsymbol{\mu})$. The parameter domain is $\mathcal{D} = [6, 20] \times [1, 3] \times [0.5, 3]$. We consider the following optimal control problem

$$\min_{y_o, u_o} J(y_o(\boldsymbol{\mu}), u_o(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\|y_o(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{L^2(\hat{\Omega}_o)} + \frac{\alpha}{2}\|u_o(\boldsymbol{\mu})\|^2_{U_o},$$

$$\text{s.t.} \begin{cases} -\dfrac{1}{\mu_1}\Delta y_o(\boldsymbol{\mu}) + x_{o2}(1 - x_{o2})\dfrac{\partial y_o(\boldsymbol{\mu})}{\partial x_{o1}} = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \\ y_o(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_D^o \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = u_o(\boldsymbol{\mu}) & \text{on } \Gamma_C^o(\boldsymbol{\mu}) \\ \dfrac{1}{\mu_1}\nabla y_o(\boldsymbol{\mu}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_N^o(\boldsymbol{\mu}), \end{cases} \tag{5.3}$$

where we impose constant Dirichlet conditions on the inlet boundary of the channel, homogeneous Neumann condition on the outlet boundary and finally a Neumann condition equal to the control function u_o on Γ_C^o. We denote with Y_o and U_o the spaces $H_0^1(\Omega_o)$ and $L^2(\Gamma_C^o)$ respectively, moreover $Q_o \equiv Y_o$. By tracing the problem back to a reference domain we obtain the parametrized formulation (2.9) where the affine decompositions (2.12) (2.13) hold with $Q_a = 1$, $Q_b = 5$, $Q_f = 1$, $Q_g = 4$.

As mentioned in §5.2, in order to avoid the time-consuming SCM algorithm, we seek for an alternative strategy to compute a lower bound of the inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. As recently proposed in [22], we consider – rather than a rigorous lower bound – a *surrogate* of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ given by an interpolation procedure. We (arbitrary and a priori) select a (possibly *small*) set of interpolation points $\Xi_\beta \subset \mathcal{D}$ and compute

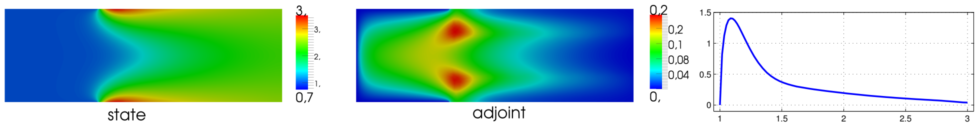

FIG. 5.10. *Test 3: representative solution for* $\boldsymbol{\mu} = (12, 2, 2.5)$. *We report the state variable* y_N *(left), the adjoint variable* p_N *(middle) and the optimal control* u_N *on* Γ_C^o *(right); thanks to the symmetry of the problem the control variable has the same values on the boundaries* $\Gamma_C \cap \{x_2 = 0\}$ *and* $\Gamma_C \cap \{x_2 = 1\}$.

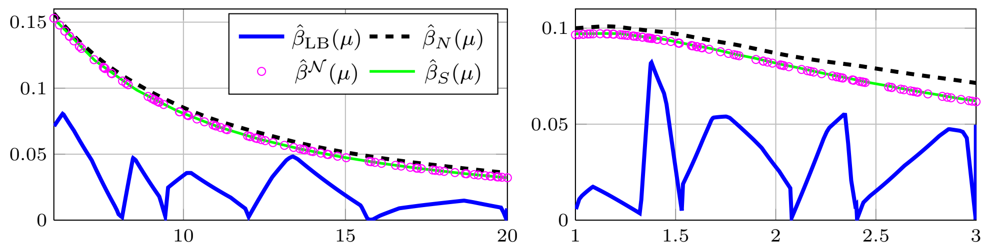

FIG. 5.11. *Test 3: comparison between lower bound and interpolant surrogate for the discrete Babuška inf-sup constant* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$. *On the left:* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of* μ_1, $(\mu_2, \mu_3) = (1.5, 3)$ *fixed; on the right:* $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ *as a function of* μ_2, $(\mu_1, \mu_3) = (9, 1)$ *fixed.*

the inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ by solving the related eigenproblem for each $\boldsymbol{\mu} \in \Xi_\beta$. Then we compute a suitable *interpolant surrogate* $\hat{\beta}_S(\boldsymbol{\mu})$ such that

$$\hat{\beta}_S(\boldsymbol{\mu}) = \hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu}), \qquad \forall \boldsymbol{\mu} \in \Xi_\beta.$$

Depending on the number of parameters and their range of variation, different interpolation methods can be employed. Here we use a simple linear interpolant and an equally spaced grid of interpolation points in the parameter space. Actually, since the parameter μ_3 does not affect the value of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$, we perform just a two dimensional interpolation with respect to the parameters μ_1 and μ_2.

We present here a first test comparing the performances of this alternative strategy with respect to the SCM algorithm. We fixed $\alpha = 0.07$ and used piecewise linear finite elements for the FE approximation, the dimension of the global FE space $\mathcal{X}^{\mathcal{N}}$ is $\mathcal{N} = 7\,156$. In Figure 5.11 we show a comparison between the lower bound for the Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ obtained using the SCM algorithm and the interpolant surrogate $\hat{\beta}_S(\boldsymbol{\mu})$; SCM takes around 1 hour to be performed, while the computation of the interpolant surrogate needs only 24 seconds using 120 sampling points in the parameter space. Furthermore, the interpolant surrogate is a much sharper approximation of the true FE inf-sup constant – despite not being a rigorous lower bound – thus resulting also in a sharper a posteriori error estimate (see Figure 5.12). For this reason, with a fixed tolerance $\varepsilon_{tol}^{rel} = 5 \cdot 10^{-4}$, the greedy algorithm selects $N_{max} = 36$ basis functions when using the lower bound given by the SCM, while only $N_{max} = 27$ basis functions are selected when employing the interpolant surrogate. A detailed comparison of the computational costs is given in Table 5.2.

Finally, we have performed a further test using a smaller regularization constant $\alpha = 8 \cdot 10^{-3}$ and a finer triangulation of the spatial domain, resulting in a global FE space $\mathcal{X}^{\mathcal{N}}$ of dimension $\mathcal{N} = 22\,792$. We use $\hat{\beta}_S(\boldsymbol{\mu})$ as surrogate for the lower bound of the FE inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$: with 120 equally distributed interpolation points we obtain a sharp approximation of $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ (see Figure 5.13a), yet requiring less than two minutes to be computed in the Offline stage (all the numerical details are given

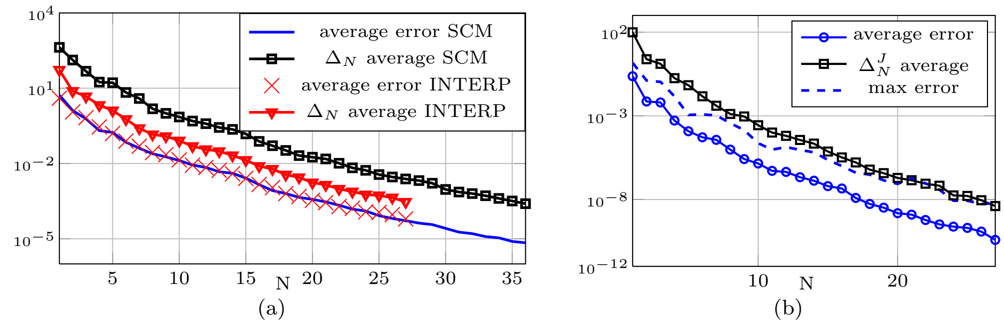

FIG. 5.12. *Test 3 ($\alpha = 0.07$). (a) True error and error estimate between the FE solution and the RB approximation: the quantities in red are obtained using the interpolant surrogate $\hat{\beta}_S(\boldsymbol{\mu})$ instead of $\hat{\beta}_{LB}(\boldsymbol{\mu})$. (b) Average true error and bound $\Delta_N^J(\boldsymbol{\mu})$ between $\mathcal{J}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\mathcal{J}_N(\boldsymbol{\mu})$ using $\hat{\beta}_S(\boldsymbol{\mu})$ in the estimate.*

TABLE 5.2
Numerical details for Test 3 ($\alpha = 0.07$). Comparison between the use of SCM algorithm and the interpolation procedure.

	SCM algorithm	Interpolation surrogate
Number of eigenvalue problems	239	120
"Lower bound" computation time	3523 s	24 s
Greedy algorithm comput. time	349 s	175 s
Number of RB functions N	36	27
Linear system size reduction	39:1	53:1

in Table 5.3). The greedy algorithm selects $N_{max} = 35$ basis functions in order to guarantee the relative error of the RB solution (with respect to the FE approximation) to be under the desired tolerance $\varepsilon_{tol}^{rel} = 5 \cdot 10^{-4}$. In Figure 5.13b we compare the a posteriori error bound $\Delta_N(\boldsymbol{\mu})$ with the true error $\|\mathsf{x}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathsf{x}_N(\boldsymbol{\mu})\|_{\mathcal{X}}$.

6. Conclusions. In this work we have developed a reduced basis framework for the efficient solution of parametrized linear-quadratic optimal control problems governed by elliptic coercive PDEs. A rigorous well-posedness analysis has been carried out by exploiting a suitable saddle-point formulation. On the other hand, the certified error bounds on the solution variables as well as on the cost functional have been obtained by recasting the problem in the form of weakly coercive problems and then applying standard arguments based on Nečas-Babuška stability theory. Finally, we have also provided a full Offline-Online decomposition strategy ensuring the Online efficiency of the method. Our numerical tests showed the possibility to obtain large computational savings (a speedup of at least two order of magnitude) in the Online stage with respect to classical high-fidelity discretization methods. In particular, the proposed error estimators demonstrate to be sharp enough to enable an efficient exploration of the parameter space through the Greedy algorithm, thus resulting in the selection of a reasonably small number of basis functions.

A possible drawback resides in the Offline stage, that demands for large computational resources. To alleviate this problem, we have provided a detailed (empirical) analysis of the computational costs required by the main operations to be performed, i.e. the computation of a lower bound for the inf-sup constant (via the SCM algorithm) and the construction of the RB spaces through the Greedy algorithm. Since the main computational effort is required by the former, we have proposed the use of a suitable interpolant surrogate instead of a rigorous lower bound. This alternative strategy is

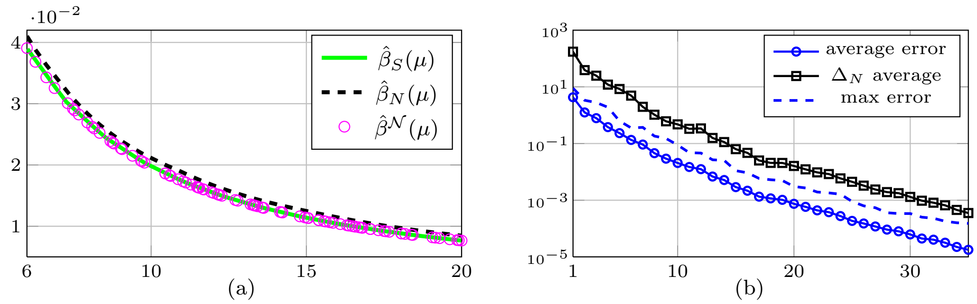

Fig. 5.13. *Test 3 ($\alpha = 8 \cdot 10^{-3}$). (a) Interpolant surrogate for the discrete Babuška inf-sup constant $\hat{\beta}^{\mathcal{N}}(\boldsymbol{\mu})$ as a function of μ_1, $(\mu_2, \mu_3) = (1.5, 3)$ fixed. (b) Average and max true errors and estimate on the solution variables.*

Table 5.3
Numerical details for Test 3 ($\alpha = 8 \cdot 10^{-3}$).

Approximation data		Computational performances	
Number of FE dof $\mathcal{N}$	22 792	Linear system size reduction	130:1
Number of parameters P	3	RB solution	2.5 ms
Affine operator components Q_B	6	Offline interpolation time	102 s
Number of RB functions N	35	Offline greedy time	860 s

signicantly more efficient, resulting in both a substantial computational saving in in the Offline stage and a sharper approximation of the true stability factor.

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REFERENCES

[1] V. Akcelik, G. Biros, O. Ghattas, J. Hill, D. Keyes, and B. Waanders, *Parallel Algorithms for PDE-Constrained Optimization*, in Parallel Processing for Scientific Computing, M.A. Heroux, P. Raghavan, and H.D. Simon, eds., Philadephia, PA, 2006, SIAM.

[2] I. Babuška, *Error-bounds for finite element method*, Numer. Math., 16 (1971), pp. 322–333.

[3] R. Becker and R. Rannacher, *An optimal control approach to a posteriori error estimation in finite element methods*, Acta Numerica, 10 (2001), pp. 1–102.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Elements Methods*, Springer-Verlag, New York, 1991.

[5] L. Dedè, *Reduced basis method and a posteriori error estimation for parametrized linear-quadratic optimal control problems*, SIAM J. Sci. Comput., 32 (2010), pp. 997–1019.

[6] ———, *Reduced basis method and error estimation for parametrized optimal control problems with control constraints*, J. Sci. Comput., 50 (2012), pp. 287–305.

[7] A.-L. Gerner and K. Veroy, *Reduced basis a posteriori error bounds for the Stokes equations in parameterized domains: A penalty approach*, Mathematical Models and Methods in Applied Sciences (M3AS), (2011).

[8] A.-L. Gerner and K. Veroy, *Certified reduced basis methods for parametrized saddle point problems*, SIAM J. Sci. Comput, (2012). Accepted.

[9] M.A. Grepl and M. Karcher, *Reduced basis a posteriori error bounds for parametrized linear-quadratic elliptic optimal control problems*, C. R. Math. Acad. Sci. Paris, 349 (2011), pp. 873 – 877.

[10] M. D. Gunzburger, *Perspectives in flow control and optimization*, SIAM, Philadelphia, 2003.

[11] M. D. Gunzburger and P. B. Bochev, *Least-Squares Finite Element Methods*, Springer, 2009.

[12] B. Haasdonk, J. Salomon, and B. Wohlmuth, *A reduced basis method for parametrized variational inequalities*, SIAM J. Numer. Anal., (2012). Accepted.

[13] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE constraints*, Springer, 2009.
[14] D.B.P. Huynh, *Reduced Basis Approximation and Application to Fracture Problems*, PhD thesis, Singapore-MIT Alliance, National University of Singapore, 2007. Available at `http://augustine.mit.edu`.
[15] D.B.P. Huynh, N.C. Nguyen, A.T. Patera, and G. Rozza, *Rapid reliable solution of the parametrized partial differential equations of continuum mechanics and transport.* Available at `http://augustine.mit.edu`, ©MIT 2008-2011.
[16] D. B. P. Huynh, D.J. Knezevic, Y. Chen, J.S. Hesthaven, and A.T. Patera, *A natural-norm successive constraint method for inf-sup lower bounds*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 1963 – 1975.
[17] K. Ito and K. Kunisch, *Lagrange Multiplier Approach to Variational Problems and Applications*, Adv. Des. Control, SIAM, 2008.
[18] K. Ito and S. Ravindran, *A reduced basis method for control problems governed by PDEs*, In W. Desch, F. Kappel and K. Kunisch eds. Control and Estimation of Distributed Parameter System, (1998), pp. 153–168.
[19] M. Kärcher, *The Reduced-Basis Method for Parametrized Linear-Quadratic Elliptic Optimal Control Problems*, master's thesis, Technische Universität München, 2011.
[20] K. Kunisch and S. Volkwein, *Proper orthogonal decomposition for optimality systems*, ESAIM Math. Modelling Numer. Anal., 42 (2008), pp. 1–23.
[21] J.L. Lions, *Optimal Control of Systems governed by Partial Differential Equations*, Springer-Verlag, Berlin Heidelberg, 1971.
[22] A. Manzoni, *Reduced models for optimal control, shape otpimization and inverse problems in haemodynamics*, PhD thesis, N. 5402, École Polytechnique Fédérale de Lausanne, 2012.
[23] J. Nečas, *Les Methodes Directes en Theorie des Equations Elliptiques*, Masson, Paris, 1967.
[24] N.C. Nguyen, K. Veroy, and A.T. Patera, *Certified real-time solution of parametrized partial differential equations*, In: Yip, S. (Ed.). Handbook of Materials Modeling, (2005), pp. 1523–1558.
[25] A. Quarteroni, G. Rozza, and A. Manzoni, *Certified reduced basis approximation for parametrized PDE and applications*, J. Math in Industry, 3 (2011).
[26] A. Quarteroni, G. Rozza, and A. Quaini, *Reduced basis methods for optimal control of advection-diffusion problems*, in Advances in Numerical Mathematics, Moscow, Russia and Houston, USA, 2007, pp. 193–216.
[27] S. S. Ravindran, *A reduced-order approach for optimal control of fluids using proper orthogonal decomposition*, Int. J. Numer. Meth. Fluids, 34 (2000), pp. 425–448.
[28] D.V. Rovas, *Reduced-basis output bound methods for parametrized partial differential equations*, PhD thesis, Massachusetts Institute of Technology, 2003.
[29] G. Rozza, D.B.P. Huynh, and A. Manzoni, *Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants*, Tech. Report 22.2010, MATHICSE. Submitted.
[30] G. Rozza, D.B.P. Huynh, and A.T. Patera, *Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations*, Arch. Comput. Methods Eng., 15 (2008), pp. 229–275.
[31] G. Rozza, A. Manzoni, and F. Negri, *Reduced strategies for PDE-constrained optimization problems in haemodynamics*, in Proceedings of the 6th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Vienna, Austria, 2012.
[32] G. Rozza and K. Veroy, *On the stability of the reduced basis method for Stokes equations in parametrized domains*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1244 – 1260.
[33] J. Schöberl and W. Zulehner, *Symmetric Indefinite Preconditioners for Saddle Point Problems with Applications to PDE-Constrained Optimization Problems*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 752–773.
[34] S. Sen, *Reduced Basis Approximation and A Posteriori Error Estimation for Non-Coercive Elliptic Problems: Application to Acoustics*, PhD thesis, Massachusetts Institute of Technology, 2007. Available at `http://augustine.mit.edu`.
[35] T. Tonn, K. Urban, and S. Volkwein, *Comparison of the reduced-basis and POD a-posteriori estimators for an elliptic linear-quadratic optimal control problem*, Math. Comput. Model. Dyn. Syst., 17 (2011), pp. 355–369.
[36] F. Tröltzsch and S. Volkwein, *POD a-posteriori error estimates for linear-quadratic optimal control problems*, Comput. Optim. Appl., 44 (2009), pp. 83–115.
[37] J. Xu and L. Zikatanov, *Some observations on Babuška and Brezzi theories*, Numerische Mathematik, 94 (2003), pp. 195–202.

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