UNIQUENESS OF FOURIER REPRESENTATIONS. CANTOR'S THEOREM

The material in the lecture is roughly based on a paper by J. Marshall Ash in the American Math. Monthly.

1. Uniqueness of Fourier representations

Up to this point we have discussed the question whether the formal Fourier series $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi ins}$ reproduces f, where

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\, dx.$$

Note that if the Fourier series

$$\sum_{n\in Z} c_n e^{2\pi inx}$$

converges uniformly to $f(x)$, then indeed the value of c_n is forced and given by the above formula. It is, however, perfectly natural to ask whether the above formula is forced even if we only assume the convergence of the Fourier series to f *in the point wise sense.* Thus could there be a different sequence of coefficients d_n, $n \in \mathbb{Z}$, such that $f(x) = \sum_{n\in Z} d_n e^{2\pi inx}$? Or alternatively, does the relation

$$\lim_{N\to\infty} \sum_{|n|\leq N} c_n e^{2\pi inx} = 0 \tag{1.1}$$

for each $x \in S^1$, without any assumption of uniformity of convergence in x, force $c_n = 0$ for all n? This is a highly non-trivial question, as evidenced by the fact that *if we weaken the hypothesis a bit* and assume (1.1) only on the complement $S^1\backslash E$ of some zero measure set E, then in fact there may be such non-vanishing coefficients, depending on deep structural properties of the set E.
Nonetheless, if $E = \emptyset$, then we indeed have the desired result:

Theorem 1.1. *(G. Cantor, 1870) The relation* (1.1) *implies $c_n = 0$ for all n.*

In the following section, we provide the proof, which is completely elementary.

2. The proof of Theorem 1.1

To simplify notation, we shall work on $[0, 2\pi]$ instead of $[0, 1]$, and replace the exponentials $e^{2\pi ins}$ by e^{ins}. The proof can be neatly divided into four parts:

(i): establish asymptotic vanishing of the c_n; in particular, the c_n are bounded.

(ii): introduce the auxiliary function $c_0\frac{x^2}{2} + \sum_{n\neq 0} c_n \frac{e^{inx}}{(in)^2}$. This series converges absolutely and hence represents a continuous function $F(x)$. Then show that the function $F(x)$ has vanishing Schwarz derivative, i. e.

$$\lim_{h\to 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = 0 \ \forall x \in [0, 2\pi].$$

(iii): Show that a continuous function with vanishing Schwarz derivative is linear.

(iv): Show that the preceding steps imply that $c_n = 0$ for all n.

2.1. **Asymptotic vanishing of c_n.** .
Here we show

Lemma 2.1. *If we have*

$$\lim_{N\to\infty}\sum_{|n|\le N} c_n e^{inx} = 0 \ \ \forall x\in[0,2\pi], \tag{2.1}$$

then $\lim_{|n|\to\infty} c_n = 0$.

Proof. By passing to differences between $s_N := \sum_{|n|\le N} c_n e^{inx}$, we immediately get

$$\lim_{n\to+\infty}[c_n e^{inx} + c_{-n}e^{-inx}] = 0$$

for all $x\in[0,2\pi]$. Write $c_{\pm n} = \operatorname{Re} c_{\pm n} + i \operatorname{Im} c_{\pm n}$. Then

$$c_n e^{inx} + c_{-n}e^{-inx} = a_n\cos nx + b_n \sin nx + i(a_n'\cos nx + b_n'\sin nx),$$

where we have

$$a_n = \sum_{\pm}\operatorname{Re} c_{\pm n},\ a_n' = \sum_{\pm}\operatorname{Im} c_{\pm n},\ b_n = \sum_{\pm}(-1)^{\frac{\pm1+1}{2}}\operatorname{Im} c_{\pm n},\ b_n' = \sum_{\pm}(-1)^{\frac{\pm1-1}{2}}\operatorname{Re} c_{\pm n}$$

In particular we have

$$a_n^2 + b_n^2 + (a_n')^2 + (b_n')^2 = 2(|c_n|^2 + |c_{-n}|^2),$$

and it suffices to show that $a_n^2 + b_n^2 \to 0$ provided

$$a_n\cos nx + b_n\sin nx \to 0 \ \ \forall x\in[0,2\pi]$$

provided $n\to+\infty$. Writing

$$\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \cos\theta_n,\ \frac{b_n}{\sqrt{a_n^2+b_n^2}} = \sin\theta_n$$

for adequate $\theta_n\in[0,2\pi]$, write

$$a_n\cos nx + b_n\sin nx = \sqrt{a_n^2+b_n^2}\cos(nx-\theta_n).$$

Arguing by contradiction, assume that there is a sequence $\{n_k\}_{k\ge1}\subset\{1,2,\dots\}$ with

$$\liminf_{k\to\infty}\sqrt{a_{n_k}^2+b_{n_k}^2} = \delta > 0.$$

By passing to a subsequence, we may assume that $n_{k+1} > 100 n_k$, say. We now construct a $x_*\in[0,2\pi]$ such that

$$\cos(n_k x_* - \theta_{n_k}) \not\to 0,$$

which then implies $a_{n_k}\cos n_k x_* + b_{n_k}\sin n_k x_* \not\to 0$, a contradiction. In fact, set $x_* = \sum_{k\ge1}\frac{\alpha_k}{n_k}$, where we inductively pick $\alpha_k\in[0,2\pi]$ in such a way that

$$n_k\sum_{k>l\ge1}(\frac{\alpha_l}{n_l}) - \theta_{n_k} + \alpha_k \in [-\frac{\pi}{4},\frac{\pi}{4}]\operatorname{mod}(2\pi)$$

Then since $0\le n_k\sum_{l>k}(\frac{\alpha_l}{n_l}) < \frac{1}{10}$, we also have

$$n_k x_* - \theta_{n_k} \in [-\frac{\pi}{3},\frac{\pi}{3}],$$

whence $\cos(n_k x_* - \theta_{n_k}) \in [\frac12, 1]$. □

2.2. **Properties of a remarkable auxiliary function.** The preceding lemma implies in particular that the sequence $\{c_n\} \subset \mathbf{C}$ is bounded. Introduce now the function

$$F(x) := c_0 \frac{x^2}{2} + \sum_{n \neq 0} c_n \frac{e^{inx}}{(in)^2};$$

Formally, the second derivative of it reproduces the original series $\sum_n c_n e^{ins}$, but of course we are not allowed to differentiate this expression term by term. However, by absolute convergence, F is a continuous function. Introduce now the Schwarz derivative

$$DF(x) := \lim_{h \to 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} =: \lim_{h \to 0} D_h F(x)$$

A priori, this might not even exist. However, we have

Lemma 2.2. *We have $DF(x) = 0$ for all $x \in [0, 2\pi]$. In fact, this is valid for all $x \in \mathbb{R}$.*

Proof. The key is to observe the relations (for any $h \neq 0$)

$$D_h e^{inx} = e^{inx} \frac{e^{inh} + e^{-inh} - 2}{h^2} = e^{inx} \big(\frac{\sin(\frac{h}{2}n)}{(-i)\frac{h}{2}}\big)^2, \; D_h(\frac{x^2}{2}) = 1.$$

Thus we get

$$D_h F = c_0 + \sum_{n \neq 0} c_n e^{inx} \big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 = \sum_{n \neq 0} c_n e^{inx} \big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big]$$

Of course here we may interchange summation and the 'operator' D, since no taking of limits is involved. Then the idea is to let $h \to 0$ and show that the limit equals $\sum_n c_n e^{inx} = 0$. Thus if we set

$$a_n = c_n e^{inx} + c_{-n} e^{-inx}, \; n \geq 1, \; a_0 = c_0,$$

we need to show that

$$\lim_{h \to 0} \sum_{n \geq 1} a_n \big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big] = 0.$$

To do this, we use the trick of 'summation by parts'. Thus let $s_n := \sum_{0 \leq k \leq n} a_k$. Then

$$\begin{aligned}\sum_{n \geq 1} a_n \big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big] &= \sum_{n \geq 1} (s_n - s_{n-1}) \big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big] \\ &= \sum_{N > n \geq 1} (s_n - s_{n-1}) \big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big] \\ &\quad - s_{N-1} \big[\big(\frac{\sin(\frac{h}{2}N)}{\frac{Nh}{2}}\big)^2 - 1\big] \\ &\quad + \sum_{n \geq N} s_n \Delta_n,\end{aligned}$$

where we have set $\Delta_n := \big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - \big(\frac{\sin(\frac{h}{2}(n+1))}{\frac{(n+1)h}{2}}\big)^2$. Observe that if we set $f(x) := \frac{\sin^2 x}{x^2}$, then

$$\Delta_n = \int_{\frac{nh}{2}}^{\frac{(n+1)h}{2}} f'(x)\, dx,$$

and further

$$\sum_{n \geq N} \big|\Delta_n\big| \leq \sum_{n \geq N} \int_{\frac{nh}{2}}^{\frac{(n+1)h}{2}} \big|f'(x)\big|\, dx \leq \big\|f'\big\|_{L^1(\mathbb{R}_+)} < \infty.$$

Given $\varepsilon > 0$, first pick N large enough such that $\big|s_n\big| < \frac{\varepsilon}{2\big\|f'\big\|_{L^1(\mathbb{R}_+)}}$ for all $n \geq N$, and bound

$$\big|\sum_{n\geq N} s_n\Delta_n\big| \leq (\sup_{n\geq N}|s_n|)\sum_{n\geq N}\Delta_n \leq \frac{\varepsilon}{2\big\|f'\big\|_{L^1(\mathbb{R}_+)}}\big\|f'\big\|_{L^1(\mathbb{R}_+)} < \frac{\varepsilon}{2}.$$

Importantly, this bound is independent of h. Then pick h small enough such that

$$\big|\sum_{N>n\geq 1}(s_n - s_{n-1})\big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big]\big| + \big|s_{N-1}\big[\big(\frac{\sin(\frac{h}{2}[N-1])}{\frac{[N-1]h}{2}}\big)^2 - 1\big]\big| < \frac{\varepsilon}{2}.$$

It follows that for h small enough, we have

$$\big|\sum_{n\geq 1} a_n\big[\big(\frac{\sin(\frac{h}{2}n)}{\frac{nh}{2}}\big)^2 - 1\big]\big| < \varepsilon,$$

as desired. □

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2.3. **Functions with vanishing Schwarz derivative.** Here we use the preceding lemmas to draw a very strong structural conclusion about the auxiliary function F:

Lemma 2.3. *Let G be a continuous function on some interval $I \subset \mathbb{R}$ which satisfies $DG(x) = 0$ for all $x \in I^\circ$. Then G is a linear function.*

Proof. We first show that if $DG > 0$ everywhere, then the function is convex. This means that for every two of the points of its graph, the straight line segment joining them is above the graph. If not, there are two points $(a, G(a))$, $(b, G(b)$ of the graph of G such that there is a point $(x, G(x))$, $x \in (a, b)$, above the straight line segment joining the points. By adding a linear function to G, we may assume $G(a) = G(b) = 0$, and thence $G(x) > 0$. In fact, we may then pick $x \in (a, b)$ to be such that G attains the maximum at x on $[a, b]$. But then if $h > 0$ is small enough such that $x \pm h \in (a, b)$, we have

$$G(x+h) + G(x-h) - 2G(x) = G(x+h) - G(x) - [G(x) - G(x-h)] \leq 0,$$

contradicting the assumption that $DG(x) > 0$.
Similarly, we see that $DG < 0$ everywhere implies that G is concave. But then if $DG = 0$ everywhere, we have $D(G \pm \varepsilon x^2) >< 0$, respectively, for any $\varepsilon > 0$, so $G \pm \varepsilon x^2$ is convex/concave, and passing to the limit $\varepsilon \to 0$, we get that G is both convex and concave, hence linear. □

2.4. **End of proof of Theorem 1.1.** The preceding lemmas imply that $F(x)$ is linear, which we now think of as a function on all of $\mathbb{R}$ (as we obviously may). Thus we may write

$$F(x) = \alpha + \beta x$$

for suitable $\alpha, \beta \in \mathbb{R}$. But then letting $x \to +\infty$, say, we get $c_0 = 0, \beta = 0$, whence

$$\sum_{n\neq 0} c_n \frac{e^{inx}}{(in)^2} = \alpha.$$

The convergence of the sum being uniform, we infer

$$c_n = \int_0^{2\pi} \alpha e^{-inx}\,dx = 0,\ n \neq 0,$$

and we are done.