Corrigé Série 7

1. The proof is basically the same, we only need to pay attention that,
   — $L^1 \subset (L^\infty)^*,\ L^\infty \subset (L^1)^*$;
   — The set of simple functions is dense in L^p space, $p \in [1. +\infty]$.

2. As the exercise 2 from the preceding problem, we introduce a smooth function $\chi(\xi)$ such that

$$\chi(\xi) = 1, \forall 1 \leq |\xi| < 2,$$
$$\chi(\xi) = 0, \forall |\xi| < 1/2, |\xi| > 4,$$

and define the function $K_\lambda \in \mathcal{S}(\mathbb{R}^n)$,

$$K_\lambda(x) = \int_{\mathbb{R}^n} e^{2\pi i x\xi}\chi(\lambda\xi)d\xi. \tag{1}$$

Since $||K_1||_{L^1}, ||K_1||_{L^2}, ||K_1||_{L^\infty}$ are uniformly bounded, we know the existence of C such that

$$||K_1||_{L^k} \leq C,\ \forall k \in [1, +\infty]. \tag{2}$$

Now for any function $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{f}(\xi)$ supp $2^j \leq |\xi| < 2^{j+1}$, we know that

$$\hat{f}(\xi)\chi(2^{-j}\xi) = \hat{f}(\xi), \tag{3}$$

thus

$$K_{2^{-j}} * f = \mathcal{F}^{-1}\Big(\hat{f}(\xi)\chi(2^{-j}\xi)\Big) = \mathcal{F}^{-1}\Big(\hat{f}(\xi)\Big) = f. \tag{4}$$

Therefore, by Young's inequality we get

$$||f||_{L^q} = ||K_{2^{-j}} * f||_{L^q} \leq ||K_{2^{-j}}||_{L^k}||f||_{L^p} \tag{5}$$

where $\frac{1}{k} = 1 + \frac{1}{q} - \frac{1}{p}$. By the same scaling argument of the exercise 2 from the preceding problem we know that

$$||K_\lambda||_{L^k} = \lambda^{\frac{n(1-k)}{k}}||K_1||_{L^k}. \tag{6}$$

$$||f||_{L^q} \leq ||K_{2^{-j}}||_{L^k}||f||_{L^p} = ||K_1||_{L^k}2^{jn(\frac{1}{p}-\frac{1}{q})}||f||_{L^p}. \tag{7}$$

The goal is to give an idea of the Sobolev embedding, for $1 \leq p < q < +\infty$ such that $\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n}$ we have that $W^{k,p}(\mathbb{R}^n) \subset W^{l,q}(\mathbb{R}^n)$. Here we only treat the case that $n(\frac{1}{p} - \frac{1}{q}) = 1$, *i.e.* $W^{1,p} \subset L^q$. Let us define the function $S_\lambda \in \mathcal{S}(\mathbb{R}^n)$,

$$S^i_\lambda(x) = \int_{\mathbb{R}^n} e^{2\pi i x\xi}\frac{\xi_i\chi(\lambda\xi)}{2\pi i|\xi|^2}d\xi, \tag{8}$$

where ξ_i is the i-th component of ξ. Thus, thanks to the fact that $2\pi i\xi_1\hat{f}(\xi) = \widehat{\partial_{x_1}f}(\xi)$, by the same argument we know that

$$\sum_{i=1}^n S^i_{2^{-j}} * (\partial_{x_i}f) = \mathcal{F}^{-1}\Big(\sum_{i=1}^n(2\pi i\xi_i\hat{f}(\xi))\frac{\xi_i\chi(2^{-j}\xi)}{2\pi i|\xi|^2}\Big) = \mathcal{F}^{-1}\Big(\hat{f}(\xi)\chi(2^{-j}\xi)\Big) = \mathcal{F}^{-1}\Big(\hat{f}(\xi)\Big) = f. \tag{9}$$

Next by Young's inequality

$$||f||_{L^q} = ||\sum_{i=1}^{n} S^i_{2^{-j}} * (\partial_{x_i} f)||_{L^q} \leq \sum_{i=1}^{n} ||S^i_{2^{-j}}||_{L^k} ||\partial_{x_i} f||_{L^p}, \tag{10}$$

and direct calculation yields, for example

$$||S^1_{2^{-j}}||_{L^k} = ||S^1_1||_{L^k} 2^{j(n(\frac{1}{p}-\frac{1}{q})-1)} = ||S^1_1||_{L^k} \leq C. \tag{11}$$

Since the constant is **independent of** j, for any $f \in W^{1,p}$ we can decompose is by $f = \sum_j f_j$ with $\hat{f}_j$ supp $2^j \leq |\xi| < 2^{j+1}$ to conclude the embedding inequality.

3. (i) For any $\varepsilon > 0$ there exists $f_n, g_n \in C_c^\infty(\mathbb{R}^n)$ such that

$$||f - f_n||_{L^p} + ||g - g_n||_{L^p} < \varepsilon. \tag{12}$$

We know that

$$\lim_{|h| \to +\infty} ||\tau_h(f_n) + g_n||^p_{L^p} = ||f_n||^p_{L^p} + ||g_n||^p_{L^p}, \tag{13}$$

thus by Minkowski inequality

$$\lim_{|h| \to +\infty} ||\tau_h(f) + g||^p_{L^p} - ||f||^p_{L^p} - ||g||^p_{L^p} < \varepsilon + \lim_{|h| \to +\infty} ||\tau_h(f - f_n) + g - g_n||^p_{L^p} < \varepsilon + \varepsilon^p. \tag{14}$$

(ii) For the bounded linear operator T, we are able to select $f \in L^p$ such that $||Tf||_{L^q} = C_0||f||_{L^p}$, where C_0 is the norm of the operator. Suppose that $q < p$ then let $|h| \to +\infty$, in the sense of limit we have

$$\begin{aligned} ||T(\tau_h f + g)||^q_{L^q} &= ||\tau_h Tf + Tg||^q_{L^q} = ||\tau_h Tf||^q_{L^q} + ||Tg||^q_{L^q} = 2C_0^q ||f||^q_{L^p}, \\ &\leq C_0^q ||\tau_h f + g||^q_{L^p} = C_0^q 2^{\frac{q}{p}} ||f||^q_{L^p}, \end{aligned}$$

hence $q \geq p$, contradiction.

4. (i) By the extension theorem of linear operators on Banach spaces, we know that the inequality holds for any $f \in L^p$. We select $f \in L^p$ such that $||\hat{f}||_{L^q} = C_0||f||_{L^p}$, where C_0 is the norm of the operator. Then we use the classical scaling trick, we know that $\widehat{f(\lambda x)}(\xi) = \lambda^{-n}\hat{f}(\frac{\xi}{\lambda})$, thus

$$||\widehat{f(\lambda x)}||_{L^q} = \lambda^{-n}\lambda^{\frac{n}{q}} ||\hat{f}||_{L^q} = \lambda^{-n}\lambda^{\frac{n}{q}} C_0 ||f||_{L^p} \tag{15}$$

On the other hand we have

$$||\widehat{f(\lambda x)}||_{L^q} \leq C_0 ||f(\lambda x)||_{L^p} = C_0 ||f(x)||_{L^p} \lambda^{-\frac{n}{p}}, \tag{16}$$

thus

$$\lambda^{n(-1+\frac{1}{q}+\frac{1}{p})} \leq 1, \forall \lambda > 0. \tag{17}$$

(ii) From (i) we know that $\frac{1}{p} + \frac{1}{q} = 1$, also we know that for $p \in [1, 2]$ the argument is true. It seems to be quite natural to prove $q \geq p$ as the preceding exercise, however, we are not able to conclude similar conditions as h tends to ∞ :

$$\lim_{h} ||\hat{f} e^{2\pi i h \cdot \xi} + \hat{g}||^q_{L^q} \neq ||\hat{f}||^q_{L^q} + ||\hat{g}||^q_{L^q}, \forall q < 2, \tag{18}$$

though which is true for $q = 2$.

Now we attempt to construct directly counterexamples. At first, we treat the case that $n = 1$. Let us introduce Shapiro polynomial, briefly speaking, we are able to find polynomials $P_n(x)$ and $Q_n(x)$ by

$$P_n(x) := \sum_{k=1}^{2^n-1} (-1)^{a_{n,k}} e^{2\pi ikx},$$
$$Q_n(x) := \sum_{k=1}^{2^n-1} (-1)^{b_{n,k}} e^{2\pi ikx},$$

satisfying

$$|P_n(x)|^2 + |Q_n(x)|^2 = 2^{n+1}. \tag{19}$$

Indeed, this can be achieved by mathematical induction, the detail of which we omit and can be found easily.

Next we select some function $\phi(x) \in \mathcal{S}$ such that $\hat{\phi}$ supp $(0, 1)$, and define

$$f_n(x) := P_n(x)\phi(x) \tag{20}$$

that satisfying

$$||f_n||_{L^p} \leq ||P_n||_{L^\infty}||\phi||_{L^p} \leq 2^{\frac{n+1}{2}}||\phi||_{L^p}. \tag{21}$$

On the other hand, direct calculation on the Fourier transformation of $e^{2\pi ikx}\phi(x)$ gives

$$||\hat{f}_n||_{L^q}^q = \int_{\mathbb{R}} |\sum_{k=1}^{2^n-1} \hat{\phi}(\xi - k)|^q d\xi = \int_{\mathbb{R}} \sum_{k=1}^{2^n-1} |\hat{\phi}(\xi - k)|^q d\xi = 2^n||\hat{\phi}||_{L^q}^q, \tag{22}$$

thus

$$||\hat{f}_n||_{L^q} \leq 2^{\frac{n}{q}}||\hat{\phi}||_{L^q}. \tag{23}$$

Therefore, by letting n tends to ∞ we get $q \geq 2$ hence $p \leq 2$.

As for multi dimensions, for $\mathbf{x} = (x_1, x') \in \mathbb{R}^n$, it suffices to define

$$f_n(\mathbf{x}) := f_n(x_1)\chi_{[0,1]^{n-1}}(x'). \tag{24}$$