Financial Markets Equilibrium with Heterogeneous Agents

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Abstract

This paper presents an equilibrium model in a pure exchange economy when investors have three possible sources of heterogeneity. Investors may differ in their beliefs, in their level of risk aversion and in their time preference rate. We study the impact of investors heterogeneity on equilibrium properties. In particular, we analyze the consumption shares, the market price of risk, the risk free rate, the bond prices at different maturities, the stock price and volatility as well as the stock's cumulative returns, and optimal portfolio strategies. We relate the heterogeneous economy with the family of associated homogeneous economies with only one class of investors. We consider cross sectional as well as long run properties.

1 Introduction

We analyze financial markets with three possible sources of heterogeneity among agents: they may differ in their beliefs, in their level of risk aversion and in their time preference rate. We analyze agents interactions, and the impact of heterogeneity at the individual level, in particular on individual consumption, individual valuations, individual portfolio holdings and risk sharing rules. At the aggregate level, we analyze properties of the market price of risk, of the risk free rate, of the bond prices, and of the stock price and volatility. We identify the channels through which heterogeneity impacts the different equilibrium characteristics and show that heterogeneity by itself permits to explain some critical features of financial markets.

Consistent with observations that "equity risk premia seem to be higher at business troughs than they are at peaks" (Campbell and Cochrane, 1999), we show that the market price of risk is always monotone decreasing in the aggregate endowment.[1] Interestingly enough, this result is very general and holds for any distribution of the parameters of risk aversion and beliefs. It is heterogeneity and its impact on the fluctuations of the relative levels of risk tolerance which generates this behavior. It should be noted that even though this result is in the spirit of the findings of Chan and Kogan (2002) or Campbell and Cochrane (1999), unlike in those papers, we do not need to impose *habit-formation* preferences (*keeping-up-with-the-Joneses* preferences).

We also identify conditions under which the risk free rate is increasing in the endowment level. This is a desirable feature of financial markets models because empirical studies have confirmed that the short term rate is a procyclical indicator of economic activity (see e.g. Friedman, 1986, Blanchard and Watson, 1986).

Our analysis of the term structure of interest rates shows that there are distinct horizons at which distinct agents drive the long term bond yield. In equilibrium, each agent effectively demands a different long run interest rate, coinciding with the interest rate in the corresponding single agent economy. The yield curve is defined stepwise, with each subinterval being associated with a given agent in the sense that the marginal rate on that subinterval corresponds to the rate in the economy populated only by this agent.[2] It is interesting to note that this subdivision of the yield curve into different segments holds in the long run even though the different agents (except one) associated to the different habitats do not survive in the long run.

This finding confirms the previously noted fact that survival and long run impact are different concepts. As far as risky assets are concerned, we also show that the long run return of these assets are impacted by nonsurviving agents and we provide an example where the agent who drives the long run discount rate is different from the agent who drives the long run risky returns and both of them are different from the surviving agent who drives the instantaneous risk free rate in the long run. In particular, the long run short term risk premium is then determined by the surviving agent while the

[1]This has been noticed as early as Dumas (1989), in the case of two agents.

[2] Interestingly enough, with more than two agents, the investment horizon is generally non-monotonic in the individual agent's interest rate, so that agents demanding a higher rate may dominate the shorter end of the yield curve.

long term risk premium (the spread between the long run risky and riskless returns) is determined by two different agents, namely those who respectively drive the long run risky and riskless returns. Heterogeneity leads then to a term structure of risk premia that is not flat and there are cases where the long term risk premium is higher than the instantaneous risk premium. In other words, the presence of heterogeneity modifies the long term relation between risk and return and introduces a distortion between the long term and the short term risk-return tradeoff.

Let us describe in more detail how heterogeneity operates. Even though the individual levels of risk aversion, optimism and patience are constant, heterogeneity leads to time and state varying levels of risk aversion, optimism and patience at the aggregate level. Indeed, the aggregate parameters can be written as a risk tolerance weighted average of the individual parameters. Since the levels of risk tolerance are time and state dependent, this generates at the aggregate level waves of risk aversion, of pessimism/optimism, and of patience. To illustrate this, let us focus on "extreme" states of the world (very high or very low level of aggregate endowment, or distant future states of the world). We find that the agent who values the most these states dominates the other agents in terms of consumption shares or relative risk tolerance. In very bad (very good, distant future) states of the world this agent corresponds to the agent with the highest individual required market price of risk[3] (lowest individual required market price of risk, lowest *survival* index[4]). The aggregate level of risk aversion (optimism, patience) is then given by the level of risk aversion (optimism, patience) of the dominating agent. For example, more pessimistic agents dominate the economy in bad states of the world when there is only beliefs heterogeneity, which leads to a pessimistic bias at the aggregate level. This can explain excess of pessimism in periods of recession without referring to irrational behavior. Analogously, more risk averse agents dominate the economy in periods of recession when there is only risk aversion heterogeneity, which leads to more risk aversion at

[3]The individual market price of risk of agent i is given by $\theta_i = \gamma_i \sigma - \delta_i$ where γ_i, δ_i and σ respectively denote the individual level of risk aversion, the individual level of optimism and the volatility of aggregate endowment. The individual (required) market price of risk reflects the agent's motives to invest in a risky asset. It increases with the level of risk aversion and with the level of pessimism.

[4]The survival index of agent i is defined by $\kappa_i \equiv \rho_i + \gamma_i(\mu - \frac{\sigma^2}{2}) + \frac{1}{2}\delta_i^2$, where $\rho_i, \gamma_i, \delta_i$, μ and σ respectively denote the individual level of time preference, risk aversion, optimism and the drift and volatility of aggregate endowment.

the aggregate level. Brunnermeier and Nagel (2008) find that the fraction of wealth that households invest into risky assets does not change with the level of wealth and conclude that the households' relative risk aversion is constant at individual level. On the other hand, fluctuating risk aversion of a representative agent, as in habit preference models (Campbell and Cochrane, 1999), would help matching aggregate data. Our results describe in which way in equilibrium with heterogeneous agents fluctuating relative risk aversion at the aggregate level is not inconsistent with constant relative risk aversion at the individual level.

In order to analyze the impact of such fluctuating aggregate parameters on the equilibrium prices (interest rates and market price of risk), let us compare our heterogeneous economy with different benchmarks corresponding to the different homogeneous economies that we would obtain if all the agents were of the same type, namely the type[5] of agent i. In our setting, the market price of risk appears as a risk tolerance weighted average of the individual market prices of risk (the prices that we would obtain in the different benchmark economies). It then fluctuates in time and states of the world between the lowest and highest individual market prices of risk. For very bad (good, distant future) states of the world, the market price of risk is given by the highest individual market price of risk (the lowest market price of risk, the market price of risk of the surviving agent). This phenomenon that operates in extreme states of the world is in fact more general leading to a market price of risk which is decreasing in the level of aggregate shocks.

Contrary to the market price of risk, the risk free rate is not a weighted average of the individual risk free rates[6]. The equilibrium risk free rate can lie outside the range bounded by the lowest individual risk free rate and by the highest individual risk free rate. However, in "extreme" states of the world, the risk free rate behaves as a risk tolerance weighted average of the individual ones and the "dominating" agent governs the risk free rate.

The equilibrium long term bond yield is given by the individual long term

[5] If we consider a model with n agents then there are n possible benchmarks. In the following, the equilibrium prices in the i^{th} benchmark economy (i.e. in the homogeneous economy where all the agents have agent i characteristics) will be called agent i individual prices.

[6] The individual risk free rate of agent i is given by $r_i = \rho_i + \gamma_i (\mu + \delta_i) - \frac{1}{2}\gamma_i (\gamma_i + 1) \sigma^2$, where $\rho_i, \gamma_i, \delta_i, \mu$ and σ respectively denote the individual level of time preference, risk aversion, the individual level of optimism, the drift and the volatility of aggregate endowment.

bond yield of the agent with the highest savings motives (lowest individual risk free rate). This is due to the fact that this agent most values the very long term bonds. A related interesting finding we obtain is that the agent who drives the long run-long term bond yield differs from the agent who drives the long run risk free rate, even though the bond yield is an average of the risk free rates. In particular, in the long run, the yield curve is driven, at one end, by the risk free rate of the agent with the lowest survival index whereas at the other end, it is driven by the risk free rate of the agent with the highest savings motives, or, equivalently, the lowest risk free rate[7]. In between, the long run yield curve is governed stepwise by different agents that maximize a trade-off between savings motives and survival. For example, when there is only heterogeneity in beliefs, one end of the long run yield curve is dominated by the most rational agent (maximization of the survival motives), the other end is dominated by the most pessimistic agent (maximization of the savings motives) and in the middle, the long run yield curve is governed, in intervals, by more and more pessimistic agents (maximization of a trade off between rationality and pessimism).

We also analyze the behavior of stock volatility, which converges to dividend volatility. In the long run, only the surviving agent is present (in terms of consumption shares or risk tolerance levels) and the stock volatility in the heterogeneous economy converges to the surviving agent's individual stock volatility, which is the dividend volatility. We show that for finite times stock volatility fluctuates between bounds determined by the maximal difference between market prices of risk associated with different agents. We get similar bounds for the optimal portfolios. When all levels of risk aversion are larger than one, then in the limit all agents determine their optimal portfolios using the market price of risk associated with the surviving agent.

We now discuss other related articles. The whole literature on equilibrium risk sharing in complete markets with heterogeneous risk preferences starts with the seminal paper by Dumas (1989). He considers an equilibrium production economy, populated by two agents with heterogeneous risk preferences and provides a detailed investigation of numerous dynamic properties of the economy, including consumption sharing rules, equilibrium optimal portfolios and properties of the interest rates. Wang (1996) investigates the Dumas (1989) two agent risk sharing problem, but in a Lucas-type exchange

[7] This result is in the spirit of Wang (1996) who considers a model with two agents that have different levels of risk aversion.

economy. Assuming that one of the agents is exactly two times more risk averse than the other, Wang derives a closed form expression for the equilibrium state price density and semi-closed form expression for the equilibrium yield curve. He shows that this simple economy is able to generate a very rich dynamics for the yield curve, whose shape changes over time with the state of the economy and is generally non-monotonic. Bhamra and Uppal (2009a) consider the Wang (1996) model, but with general risk aversion and show that heterogeneity may lead to excess stock price volatility. Bhamra and Uppal (2009b) extend the analysis of Bhamra and Uppal (2009a) and derive expressions for the consumption sharing rule and equilibrium characteristics in the form of infinite series.

Chan and Kogan (2002) consider an extension of the Wang (1996) model, but with keeping-up-with-the-Joneses preferences and a continuum of agents with heterogeneous risk aversions. They provide extensive numerical analysis of the equilibrium and show that the model is able to generate equilibrium moments of asset prices and returns that coincide with those observed empirically. Xiouros and Zapatero (2010) derive a closed form expression for the equilibrium state price density in the Chan and Kogan (2002) model. This allows them to better understand the precise impact of preferences heterogeneity on equilibrium dynamics. Cvitanić and Malamud (2010) study how long run risk sharing depends on the presence of multiple agents with different levels of risk aversion.

Another quite large direction of the complete market risk sharing literature concentrates on the equilibrium effects of heterogeneous beliefs. With CRRA agents differing only in their beliefs, the equilibrium state price density can be derived in closed form and therefore many equilibrium properties can be analyzed in detail. See, e.g., Basak (2000, 2005), Jouini and Napp (2007, 2010), Jouini et al. (2010) and Xiong and Yan (2010). Several papers study the market selection hypothesis, stating that irrational agents cannot survive in a competitive market, as they will constantly lose money betting on the realization of very unlikely states of the economy. For example, Sandroni (2000) and Blume and Easley (2006) show that this hypothesis is indeed true in the framework of a general, complete market, discrete time economy with bounded growth (implying that the aggregate endowment is bounded away from zero and infinity). Namely, they show that only the agents with the most rational (correct) beliefs will survive in the long run, and the consumption share of irrational agents (i.e., those whose beliefs are less correct or less efficiently updated) will go to zero and they will vanish in the long run. In particular, survival in bounded economies is independent of risk preferences. Yan (2008) shows the market selection hypothesis is still true even with unbounded growth, but survival does depend on risk aversion. He analyzes the same model as the one studied in the current paper: standard exchange economy populated by an arbitrary number of agents with heterogeneous risk aversion, discount rates and beliefs, and the aggregate endowment following a geometric Brownian motion. Yan shows that only the agent with the smallest survival index survives in the long run, but the survival index depends on both risk preferences and beliefs. However, he also shows that this market selection mechanism is very slow and it may take a very long time for an irrational agent to disappear. Berrada (2010) comes to a similar conclusion in a two-agent setting with heterogeneous beliefs and learning. Fedyk, Heyerdahl-Larsen and Walden (2010) extend Yan's (2008) model by allowing for many assets. They show that errors made by an irrational agent due to his incorrect beliefs about the multiple stocks in the economy tend to aggregate and lead to a dramatic increase in the speed of the market selection mechanism.

Given the above mentioned survival results, it is natural to ask whether the long run equilibrium quantities converge to those determined by the single surviving agent. Yan (2008) shows that this is indeed true for both the market price of risk and the interest rate. It is also possible to show that the same is true in the Blume and Easley model. Surprisingly, Kogan, Ross, Wang and Westerfield (2006), henceforth, KRWW (2006), show that this convergence result is not anymore true for unbounded growth economies and assets with long maturity payoffs. KRWW (2006) consider a continuous time economy, populated by two CRRA agents with identical risk aversion and heterogeneous beliefs, maximizing utility from terminal wealth; they show that, as the horizon of the economy tends to infinity, the agent with incorrect beliefs (i.e., the irrational agent) does not survive. Nevertheless, he still may have a significant equilibrium impact on the stock price for a large fraction of the economy's horizon. Cvitanić and Malamud (2011) extend the results of KRWW (2006) to a multi-agent setting with heterogeneity in both preferences and beliefs and show that, with more than two agents, a new related phenomenon arises: irrational agents who neither survive nor have any price impact may still have a significant impact on other agents' equilibrium optimal portfolios.

However, as both KRWW (2006) and Cvitanić and Malamud (2011) note, these results are limited to economies with no intermediate consumption and agents maximizing utility only from terminal wealth at a finite horizon T. Kogan, Ross, Wang and Westerfield (2008), henceforth, KRWW (2008), study the link between survival and price impact in the presence of intermediate consumption and allow for general utilities with unbounded relative risk aversion and a general dividend process. They show that in order to have non-surviving agents who impact the long-run equilibrium state prices, it is necessary to assume utilities with an unbounded relative risk aversion that grows sufficiently fast at infinity. The assumption of unbounded risk aversion is strong, especially given that most existing models in finance assume that the agents have CRRA preferences. Our results on the long run behavior of the yield curve show that the phenomenon of decoupling price impact and survival can also be present in models with intermediate consumption: non-surviving agents may still have a significant impact on long maturity bond yields and long run cumulative stock returns. This result complements the results of KRWW (2008): even though, with bounded (constant) relative risk aversion, long run state price densities converge to those determined by the single surviving agent, long run bond prices do not converge to those when the maturity is sufficiently long.

Other papers with non-CRRA utilities include Hara et al. (2007), Berrada et al. (2007) and Cvitanić and Malamud (2009).

In Section 1 we present the model, we study homogeneous equilibria in Section 3, analyze the equilibrium market price of risk and risk free rate in Section 4, the equilibrium drift, volatility, risky asset returns and optimal portfolios in Section 5, survival issues in Section 6, and bond prices and the term structure of interest rates in Section 7, conclude with Section 8, and provide the proofs in Appendix.

2 The Model

We consider a continuous-time Arrow-Debreu economy with an infinite horizon, in which heterogeneous agents maximize their expected utility from future consumption.

Uncertainty is described by a one-dimensional, standard Brownian motion $W_t,\ t \in [0\,,\infty)$ on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the augmented filtration generated by W_t. There is a single consumption good and we denote by D the aggregate dividend or endowment process. We make the assumption that D satisfies the following stochastic differential equation

$$dD_t = \mu D_t dt + \sigma D_t dW_t \qquad D_0 = 1$$

where the mean growth rate μ and the volatility σ are constants.

There are N (types of) agents indexed by $i = 1, ..., N$. Agents have different expectations about the future of the economy. More precisely, agents disagree about the mean growth rate. We denote by μ_i the mean growth rate anticipated by agent i. Letting

$$\delta_i \equiv \frac{\mu_i - \mu}{\sigma}$$

denote agent i's error in her perception of the growth of the economy normalized by its risk[8], we introduce the probability measure P^i, defined by its density $Z_{it} = e^{\delta_i W_t - \frac{1}{2}\delta_i^2 t}$. From agent i point of view, the aggregate endowment process satisfies the following stochastic differential equation

$$dD_t = \mu_i D_t dt + \sigma D_t dW_t^i \qquad D_0 = 1$$

where, by Girsanov Theorem, $W_t^i \equiv W_t - \delta_i t$ is a Brownian motion with respect to P^i. The fact that agents agree on the volatility parameter is implied by the assumption that all individual probabilities P^i are equivalent to P *for every finite* t. This assumption is quite natural[9]. Moreover, as already noticed by Basak (2000), or Yan (2008), this parametrization is consistent with the insight from Merton (1980) that the expected return is harder to estimate than the variance. Note however that, even though P and P^i are equivalent when restricted to $\mathcal{F}_t$ for any $t < \infty$, they are *mutually singular* on $\mathcal{F}_\infty$. Indeed, since W_t has a drift δ_i under P^i and the drift is 0 under P, the strong law of large numbers for Brownian motion implies that, when $t \to \infty$:

$$W_t/t \quad \to \quad 0 \qquad\qquad P - a.s. \tag{1}$$

$$W_t/t \quad \to \quad \delta_i \qquad\qquad P^i - a.s. \tag{2}$$

[8]The parameter δ_i also represents the difference between agent i's perceived Sharpe ratio and the true one.

[9]Note that if P^i were absolutely continuous with respect to P and not equivalent, and if there existed an event A with a positive probability for some agent and a zero probability for another one, equilibrium could not be reached because either the demand of the first agent would be $+\infty$ or the demand of the second agent would be $-\infty$.

This means that the measure P is supported on those paths of W that stay equal to zero on average, whereas P^i is supported on those paths of W that increase (or, decrease if $\delta_i < 0$) as $\delta_i t$ on average. Clearly, these two sets of paths do not intersect and therefore the measures P and P^i are mutually singular on $\mathcal{F}_\infty$.

Note also that agents are persistent in their mistakes as in e.g. Kogan et al. (2006) or Yan (2008, 2010). This setting is the most simple and natural extension to the case with disagreement of a standard rational model where all agents know that the true growth rate is a constant μ . The restriction implied by such a modeling is that agents systematically overestimate (optimism) or underestimate (pessimism) the growth rate. This restriction is consistent with the interpretation of the bias on the beliefs as a behavioral bias characterizing the behavior of individuals towards risk, like the individual distortions of the underlying probability distributions, from behavioral decision theory literature. With such an interpretation, an individual is more or less pessimistic in the same way as she is more or less risk tolerant or impatient[10]. The choice of constant parameters can also model "tastes for assets" as in e.g. Fama and French (2007). In this case, a positive δ would correspond to the agents who like the asset whose dividends are given by the aggregate endowment process D and a negative δ to the agents who dislike that asset. Furthermore, even though assuming constant δ^i's may seem incompatible with learning, the case with constant parameters may be seen as an approximation of the situation where all the parameters are stochastic and where learning is regularly offset by new shocks on the drift μ. Indeed, as underlined by Acemoglu et al. (2009) a small amount of uncertainty (about the model characteristics) may lead to a substantial (non-vanishing) amount of long run disagreement: long run disagreement is discontinuous at certainty. Disagreement is then the rule rather than the exception. When rationality of beliefs is defined relative to what is learnable from the data rather than to some model, rational agents may exhibit drastic differences in beliefs even when they have the same information and even if they observe infinite se-

[10]If the bias corresponds to a behavioral bias having decision theoretical foundations, then it is consistent to suppose that the bias is persistent: agents remain optimistic or pessimistic. Our notion of optimism/pessimism coincides in our setting with the notions of optimism/pessimism adopted by e.g., Yaari (1987), Chateauneuf and Cohen (1994) and Diecidue and Wakker (2001). Chateauneuf and Cohen (1994) relate it to the notion of First Stochastic Dominance, while Yaari (1987) and Diecidue and Wakker (2001) relate it to the notion of Monotone Likelihood Ratio. These notions coincide in our setting.

quences of signals. Persistent disagreement is also obtained in models with learning when agents exhibit overconfidence as in Scheinkman and Xiong (2003), Xiong and Yan (2010) or Dumas et al. (2009).

Agent i's utility function is given by $u_i(c) = \frac{c^{1-\gamma_i}-1}{1-\gamma_i}$ for $\gamma_i > 0$, where γ_i is the relative risk aversion coefficient. In the following, we let $b_i \equiv \frac{1}{\gamma_i}$ denote the relative risk tolerance of agent i. Agent i's time preference rate is denoted by ρ_i.

Agent i's utility for a given consumption stream (c_t) is then given by

$$E^{P^i}\left[\int_0^\infty e^{-\rho_i t}\frac{c_t^{1-\gamma_i}-1}{1-\gamma_i}dt\right]$$

where E^{P^i} denotes the expectation operator from agent i's perspective. There are then in our setting three possible sources of heterogeneity among agents: heterogeneity in beliefs, heterogeneity in time preference rates and heterogeneity in risk aversion levels.

Agents have endowments denoted by $\left(e^{*^i}\right)$ with $\sum_{i=1}^{N} e^{*^i} = D$. We assume that markets are complete which means that all Arrow-Debreu securities can be traded. A state price density (or stochastic discount factor) M is a positive process such that $M(t,\omega)$ corresponds to the price of the asset that pays one dollar at date t and in state ω. For a given state price density M, agent i's intertemporal optimization program is given by

$$(O_{iM}) : \max_c \left\{E^{P^i}\left[\int_0^\infty e^{-\rho_i t}u_i(c_t)\,dt\right] \mid E\left[\int_0^\infty M_t\left(c_t - e_t^{*^i}\right)dt\right] \leq 0\right\}.$$

We adopt the usual definition of equilibrium.

Definition 2.1 *An equilibrium consists of a state price density M and consumption processes (c_{it}) such that each consumption process (c_{it}) solves agent i's optimization program (O_{iM}) and markets clear, i.e.,* $\sum_{i=1}^{N} c_{it} = D_t$.

We assume that such an equilibrium exists and in the following we let M (resp. c_{it}) denote the equilibrium state price density (resp. the equilibrium consumption processes).

In order to deal with asset pricing issues, we suppose that agents can continuously trade in a riskless asset and in a risky stocks[11]. We let S^0 denote the riskless asset price process with dynamics $dS_t^0 = r_t S_t^0 dt$, the parameter r denoting the risk free rate. Since there is only one source of risk, all risky assets have the same instantaneous Sharpe ratio and it suffices to focus on one specific risky asset. We consider the asset S whose dividend process is given by the total endowment of the economy and we denote respectively by μ_S and σ_S its drift and volatility. We let

$$\theta \equiv \frac{\mu_S + D\,S^{-1} - r}{\sigma_S}$$

denote the asset's Sharpe ratio or equivalently the market price of risk. The parameters r, μ_S and σ_S are to be determined endogenously in equilibrium.

We let $B(t,T)$ denote the price at time t of the pure-discount bond price delivering one dollar at time T, i.e.,

$$B(t,T) \equiv \frac{1}{M_t} E_t\left[M_T\right].$$

We also introduce the average discount rate ("yield") $Y(t,T)$ between time t and time T defined by

$$Y(t,T) \equiv -\frac{1}{T-t} \log\, B(t,T).$$

In order to deal with long run issues, we recall the following terminology. We say[12] that two processes X_t and Y_t are asymptotically equivalent if $\lim_{t\to\infty} \frac{X_t}{Y_t} = 1$ P-a.s. which we denote by $X_t \sim Y_t$ P-a.s. We say that a process X_t asymptotically dominates a process Y_t under P if $\lim_{t\to\infty} \frac{Y_t}{X_t} = 0$ P-a.s.

The quantity $\frac{c_{it}}{D_t}$ represents the consumption share of agent i at time t (in equilibrium). We also introduce the quantity

$$\omega_{it} \equiv \frac{b_i c_{it}}{\sum_{j=1}^N b_j c_{jt}} \tag{3}$$

[11]We refer to Duffie and Huang (1985) and to Riedel (2001) to show that our Arrow-Debreu equilibrium can be implemented by continuous trading of such long-lived securities.

[12]As in e.g. Kogan et al. (2006).

which represents the relative level of absolute risk tolerance[13] of agent i at time t, and plays an important part in describing the equilibrium (see, e.g. Jouini and Napp, 2007).

3 Equilibrium in homogeneous economies

We start by considering the equilibrium characteristics that would prevail in an economy made of agent i only, or that would prevail in our economy if all the initial endowment were concentrated on agent i.

We denote by M_i the equilibrium state price density in an economy with only agent i. By the first order conditions in the homogeneous economies, we have

$$M_{it} = e^{-\rho_i t} Z_{it} D_t^{-\gamma_i} \;=\; e^{-\left(\rho_i+\gamma_i\left(\mu-\frac{\sigma^2}{2}\right)+\frac{1}{2}\delta_i^2\right)t \;+\; (\delta_i-\gamma_i\sigma)W_t}.$$

The market price of risk $\theta_i \equiv \frac{\mu_S(t)+D_t S_t^{-1}-r_{it}}{\sigma_S(t)}$, the risk free rate r_i, the survival index κ_i (Yan, 2008), the stock's drift μ_{iS} and volatility σ_{iS} are respectively given[14] by

$$\begin{aligned} &\theta_i = (\gamma_i\sigma - \delta_i),\ r_i = \rho_i + \gamma_i\mu_i - \frac{1}{2}\gamma_i\left(\gamma_i+1\right)\sigma^2, \\ &\kappa_i \equiv \rho_i + \gamma_i(\mu - \frac{\sigma^2}{2}) + \frac{1}{2}\delta_i^2, \\ \mu_{iS} &= r_i + \sigma\theta_i \text{ and } \sigma_{iS} = \sigma. \end{aligned}$$

The risk free rate represents the agent's savings motives. The savings motives increase with pessimism and with patience. We index by I_0 the agent with the highest savings motives, i.e., such that $r_{I_0} \equiv \inf_i r_i$.

The market price of risk represents the agent's motives to invest in the risky asset. It increases with pessimism and with risk aversion. We index by $I_{\theta_{\max}}$ ($I_{\theta_{\min}}$) the agent with the highest (lowest) market price of risk.

With these notations we have $M_{it} = e^{-\kappa_i t \; -\theta_i W_t}$ and the survival index satisfies $\kappa_i = -\frac{1}{t}E\left[\log M_{it}\right]$. It can then be interpreted as the growth rate of

[13] The relative level of absolute risk tolerance of agent i at time t is given by $-\frac{u_i'}{u_i''}\left(c_{it}\right)\left[\sum_{j=1}^N -\frac{u_i'}{u_i''}\left(c_{it}\right)\right]^{-1}$.

[14] Letting μ_{M_i} and σ_{M_i} respectively denote the drift and volatility of the state price density M_i, the market price of risk and the risk free rate satisfy $r_{it} = -\mu_{M_i}(t)$ and $\theta_{it} = -\sigma_{M_i}(t)$.

the state price density M_i. It decreases with patience, rationality, and risk aversion when $\mu \geq \frac{\sigma^2}{2}$. The survival index differs from the risk free rate by an Itô's term, more precisely we have $r_i = \kappa_i - \frac{1}{2}\theta_i^2$. We index by I_K the agent with the lowest survival index.

We make the assumption that each of the criteria is minimal (or maximal) for one agent only, i.e., that I_0, I_K, $I_{\theta_{\min}}$ and $I_{\theta_{\max}}$ are well defined and unique. If there is only heterogeneity in time preference rates, the agent with the lowest survival index is also the agent with the highest savings motives (agent I_K coincides with agent I_0) and is the most patient agent. If there is only heterogeneity in beliefs, agent I_K is the most rational agent and differs from the agent with the highest savings motives who is the most pessimistic agent. If there is only heterogeneity in risk aversion and if $\mu > \frac{\sigma^2}{2}$, agent I_K is the least risk averse agent.

The next proposition can be readily verified. It sums up the main results about the equilibrium characteristics in the homogeneous economies.

Proposition 3.1 *Considering the homogeneous economies made of agent i only, the following properties hold almost surely under* P :

- *The state price density of the agent with the lowest (resp. highest) market price of risk dominates the other state price densities for positive (resp. negative) large values of* W*, i.e.* $\lim_{W_t \to +\infty} \frac{M_i(t,W_t)}{M_{I_{\theta_{\min}}}(t,W_t)} = 0$ *for all* $i \neq I_{\theta_{\min}}$ *and* $\lim_{W_t \to -\infty} \frac{M_i(t,W_t)}{M_{I_{\theta_{\max}}}(t,W_t)} = 0$ *for all* $i \neq I_{\theta_{\max}}$.

- *The state price density of the agent with the lowest survival index asymptotically dominates the other state price densities, i.e.,* $\lim_{t \to \infty} \frac{M_{it}}{M_{I_K t}} = 0$ *for all* $i \neq I_K$.

- *The savings motives drive the risk free rate and the bond price. We have, for all* (t,T) *,* $B_i(t,T) = e^{-r_i(T-t)}$ *and* $Y_i(t,T) = r_i$*. The bond price of the agent with the highest savings motives asymptotically dominates the other bond prices, i.e.,* $\lim_{T \to +\infty} \frac{B_i(t,T)}{B_{I_0}(t,T)} = 0$ *for all* $i \neq I_0$.

In particular, different agents dominate different prices (associated to their respective economies); agent I_0 dominates the long run bond prices, agent I_K asymptotically dominates Arrow-Debreu prices, agent $I_{\theta_{\min}}$ dominates the prices of the Arrow-Debreu assets associated to the very good states of the world and agent $I_{\theta_{\max}}$ dominates the prices of the Arrow-Debreu assets associated to the very bad states of the world.

We next consider how these comparative results among homogeneous economies can help us to better understand the equilibrium properties of heterogeneous economies.

4 Risk free rate and market price of risk

In this section we shall see that, in the heterogeneous economy, the interest rate has complex stochastic dynamics, as does the market price of risk, although the latter is somewhat simpler to study than the former, as it is a weighted average of the market prices of risk in homogeneous economies.

If we denote by μ_M and σ_M the drift and the volatility of the state price density process M, it is easy to obtain as in the standard setting that the short term rate r_t and the market price of risk $\theta_t \equiv \frac{\mu_S(t)+D_t S_t^{-1}-r_t}{\sigma_S(t)}$ are respectively given by $r_t = -\mu_M(t)$ and $\theta_t = -\sigma_M(t)$.

The next proposition gives us the expression of the risk free rate and of the market price of risk in our heterogeneous setting.[15]

Proposition 4.1 *The market price of risk is given by*

$$\theta_t = \sum_{i=1}^{N} \omega_{it}\theta_i,$$

and the risk free rate is given by[16]

$$r_t = \sum_{i=1}^{N} \omega_{it} r_i \; + \; \frac{1}{2}\sum_{i=1}^{N} (1-b_i)\,(\theta_i - \theta_t)^2 \omega_{it}$$

where we recall that $\omega_{jt} \equiv \frac{b_j c_{jt}}{\sum_k b_k c_{kt}}$*, and where* $\theta_i \equiv \gamma_i \sigma - \delta_i$ *and* $r_i \equiv \rho_i + \gamma_i \mu_i - \frac{1}{2}\gamma_i(\gamma_i+1)\sigma^2$ *respectively denote the market price of risk and the risk free rate in the economy populated by agent* i *only.*

[15] Detemple and Murthy (1997, Proposition 5) gives the expression of the risk free rate and of the market price of risk in a model with portfolio constraints, heterogeneous beliefs, heterogeneous risk aversion levels and homogeneous time preference rates.

[16] We thank Roman Muraviev for indicating this simple expression for the risk free rate.

The risk free rate and the market price of risk fluctuate in time and state of the world and these fluctuations are directly related to the fluctuations of the relative levels of risk tolerance ω_{it}. In particular, when one agent dominates the others in terms of risk tolerance then her individual market price of risk (risk free rate) dominates the equilibrium market price of risk (risk free rate). The market price of risk is a weighted average of the market prices of risk in the homogeneous economies. It fluctuates between the two bounds which are the lowest and the highest market price of risk in the different homogeneous economies. The risk free rate differs from a weighted average of the homogeneous risk free rates and, in particular, can be lower than the lowest risk free rate, or higher than the highest risk free rate. For instance, consider the case where only δ_i is heterogeneous. It is easy to see that r_t is then given by

$$r_t = E^{\omega_t}\left[r_i\right] + \frac{1}{2}(1-b)Var^{\omega_t}\left[\delta_i\right]$$

where E^{ω_t} and Var^{ω_t} are respectively the expectation and the variance operators associated with the weights ω_{it}. In particular, in the case $N = 2$, $e^{*^1} = e^{*^2}$ and $\delta_1 = -\delta_2$, r_0 lies in $[r_1, r_2]$ if and only if $|1-b|\,\delta \leq 2\gamma\sigma$. In general, we have the following result.

Corollary 4.1 • *The market price of risk satisfies*

$$\min_i \theta_i \leq \theta_t \leq \max_i \theta_i.$$

In addition, we have

$$\lim_{W_t \to +\infty} \theta\left(t, W_t\right) = \min_i \theta_i = \theta_{I_{\theta_{\min}}}, \lim_{W_t \to -\infty} \theta\left(t, W_t\right) = \max_i \theta_i = \theta_{I_{\theta_{\max}}}$$

and the long run market price of risk is given by $\lim_{t\to\infty} \theta_t = \theta_{I_K}$.

- *The risk free rate satisfies*

$$r_t \leq \sum_{i=1}^{N} \omega_{it} r_i \leq \max_i r_i \qquad \text{if } \gamma_i \leq 1 \text{ for all } i,$$

$$r_t \geq \sum_{i=1}^{N} \omega_{it} r_i \geq \min_i r_i \equiv r_{I_0} \qquad \text{if } \gamma_i \geq 1 \text{ for all } i.$$

In addition, we have

$$\lim_{W_t \to +\infty} r(t, W_t) = r_{I_{\theta_{\min}}} \ , \ \lim_{W_t \to -\infty} r(t, W_t) = r_{I_{\theta_{\max}}}$$

almost surely under P *and the long run risk free rate is given by* $\lim_{t\to\infty} r_t = r_{I_K}$.

The result on the long run risk free rate can be seen as the generalization of Yan (2008, Corollary 1) to the case with heterogeneous risk aversions and time preference rates. The long run behavior of the risk free rate and of the market price of risk is driven by the agent with the lowest survival index only. She is the only *surviving* agent (in the sense of the consumption share, or of the relative level of risk tolerance), hence is the only one to have a long run impact the instantaneous risk free rate and market price of risk.

Analogously, only the agent with the lowest (resp. highest) market price of risk impacts the behavior of the risk free rate and of the market price of risk in the heterogeneous economy for very high (resp. very low) values of W_t.[17]. In particular, the market price of risk in the heterogeneous economy reaches the two bounds in very good and very bad states of the world. It is minimal in very good states of the world, and maximal in very bad states of the world.

The next corollary shows how the market price of risk fluctuates with aggregate endowment.

Corollary 4.2 *The market price of risk* $\theta_t = \theta(t, W_t)$ *is monotone decreasing in* W_t *for any parameters of the model. That is, the market price of risk is always monotone decreasing in the level of the aggregate endowment.*

This property generalizes to the whole range of possible levels of aggregate endowment the fact that the market price of risk is governed by the agents with low market prices of risk for high levels of aggregate endowment and by the agents with high market prices of risk for low levels of aggregate endowment. Roughly speaking, an increase in aggregate endowment increases the weight of the agents that are more exposed to risk and thus of the agents that have lower market prices of risk. Note that this monotonicity

[17]That agent is the only agent present in the economy (in the sense of the consumption shares or of the relative levels of risk tolerance) in those states; see Proposition 6.1 and Corollary 6.1 below.

property is consistent with the observed variations of the equity premium. Indeed, there is evidence that the equity premium is time varying and as noted by, e.g., Campbell and Cochrane (1999) "equity risk premia seem to be higher at business cycles troughs than they are at peaks". This result generalizes the result obtained by Jouini and Napp (2010) in the specific setting of agents who only differ in their beliefs and who are on average rational. It is quite striking to obtain the monotonicity result for any distribution of the characteristics (risk aversion level, beliefs, time preference rates). It is heterogeneity and its impact on the fluctuations of the relative levels of risk tolerance ω_{it} which generates this behavior.

We also get monotonicity results for the risk free rate in the case of homogeneous risk aversion.

Corollary 4.3 *If risk aversion is homogeneous, that is $b_i = b$ for all i, then*

- *if the sequences $-\sigma(\delta_i) + 0.5(b-1)(\delta_i^2) - \rho_i$ and (δ_i) are anti-comonotone, then r_t is monotone increasing in D_t,*
- *if the sequences $-\sigma(\delta_i) + 0.5(b-1)(\delta_i^2) - \rho_i$ and (δ_i) are comonotone, then r_t is monotone decreasing in D_t.*

For instance, if time preference parameters are also homogeneous, and if agents have logarithmic utility functions, we immediately get that the risk free rate is increasing in the aggregate endowment. For general utility functions, we still obtain the monotonicity result as long as agents are not biased in their beliefs. These results remain valid if time preference rates ρ_i are no longer homogeneous but comonotone with the beliefs δ_i. These results are consistent with observed behavior: empirical studies have confirmed that the short term rate is a procyclical indicator of economic activity (see e.g. Friedman, 1986, Blanchard and Watson, 1986).

5 Stock price dynamics; optimal portfolios

We have determined the expression for the market price of risk θ_t and for the risk free rate r_t and analyzed their long run properties in Section 4. We now analyze the expression of the drift $\mu_S(t)$ and volatility $\sigma_S(t)$ of the stock price and their long run properties. We also analyze the long run properties of the price-dividend ratio and of the cumulative returns. In particular, are they given by the equilibrium quantities in the economy consisting of the surviving agent only?

We recall that in the homogeneous economies the volatility is a constant given by σ . In the homogeneous economy populated only by agent i it is easy to obtain that the stock price is finite if and only if, as in Yan (2008), $\rho_i + (\gamma_i - 1)\mu_i - \frac{1}{2}\gamma_i(\gamma_i - 1)\sigma^2 > 0$. The stock price-dividend ratio at time t is then constant and given by

$$\left(\frac{S}{D}\right)_i \equiv E_t\left[\int_t^\infty \frac{M_{i\tau} D_\tau}{M_{it}\, D_t}\, d\tau\right] = \left[\rho_i + (\gamma_i - 1)\mu_i - \frac{1}{2}\gamma_i(\gamma_i - 1)\sigma^2\right]^{-1}.$$

The cumulative expected return on rolling all the money in stock between time t and time T is then given by

$$R_i(t, T) \equiv E_t\left[\frac{S_{iT}}{S_{it}} e^{\int_t^T \left(\frac{D}{S}\right)_{i\tau} d\tau}\right] = e^{\left[\mu + \left[\left(\frac{S}{D}\right)_i\right]^{-1}\right](T-t)}$$

and the associated yield curve,

$$T \to \frac{1}{T-t} \log R_i(t, T) = \mu + \rho_i + (\gamma_i - 1)\mu_i - \frac{1}{2}\gamma_i(\gamma_i - 1)\sigma^2$$

is flat and the same for all t. We next consider what happens in the presence of heterogeneity.

5.1 Volatility and price-dividend ratio

In our heterogeneous economy, we obtain the following results on the volatility and the price-dividend ratio. Recall that $\theta_i = \gamma_i\sigma - \delta_i$.

Proposition 5.1 *1. The volatility parameter of the stock price is given by*

$$\sigma_S(t) = \sigma + \frac{E_t\left[\int_t^\infty (\theta_t - \theta_\tau) M_\tau D_\tau d\tau\right]}{E_t\left[\int_t^\infty M_\tau D_\tau\, d\tau\right]}.$$

In particular,

$$\sigma + \min_i \theta_i - \max_i \theta_i \leq \sigma_t^S \leq \sigma + \max_i \theta_i - \min_i \theta_i,$$

2. *The long run stock price volatility satisfies*

$$\lim_{t\to\infty} \sigma_S(t) = \sigma$$

almost surely under P.

3. *The long run price-dividend ratio satisfies*

$$\lim_{t\to\infty} \frac{S_t}{D_t} = \left(\frac{S}{D}\right)_{I_K}$$

almost surely under P.

4. *Suppose risk aversion is homogeneous. Then,*

- *If $0.5(1-b)\delta_i^2 + \rho_i - \sigma\delta_i$ is anti-comonotone with δ_i then $\frac{S_t}{D_t}$ is monotone increasing in W_t and the excess volatility is positive, i.e. $\sigma_S(t) \geq \sigma$.*
- *If $0.5(1-b)\delta_i^2 + \rho_i - \sigma\delta_i$ is comonotone with δ_i then $\frac{S_t}{D_t}$ is monotone decreasing in W_t and the excess volatility is negative, i.e. $\sigma_S(t) \leq \sigma$.*

The volatility is not a constant as in the standard setting, due to the stochastic market price of risk. It can fluctuate in time and state of the world. In particular, as in Bhamra and Uppal (2009a), agent's heterogeneity may lead to excessive volatility. Point 4. shows that the same kind of conclusions might be obtained when beliefs and time preference rates are heterogeneous. For instance, with homogeneous time preference parameters[18] and log utility functions, beliefs heterogeneity leads to an increase of stock volatility. The same result applies for $b \leq 1$ if all agents are pessimistic. The previous proposition also gives us the range in which volatility fluctuates. As far as long run properties are concerned, we obtain a positive answer to the question raised at the beginning of the section: only the surviving agent (i.e., the agent with the lowest survival index) has an impact on the long run volatility and price-dividend ratio.

However, we now show that even though non-surviving agents do not have an impact on the long run volatility and price-dividend ratio, they may have an impact on the long run returns.

[18]This condition may be replaced by assuming that time preference parameters and beliefs are anti-comonotone across the agents.

5.2 Cumulative returns

The cumulative expected return on rolling all the money in the stock between time t and T is given by

$$R(t,T) = E_t\left[\frac{S_T}{S_t}e^{\int_t^T D_s S_s^{-1} ds}\right] \tag{4}$$

$$= E_t^{(1)}\left[\frac{S_T}{D_T}e^{\int_t^T D_s S_s^{-1} ds}\right] E_t\left[D_T\right]\frac{D_t}{S_t} \tag{5}$$

where $P_T^{(1)}$ is the probability measure on $\mathcal{F}_T$ whose density with respect to the restriction P_T of P on $\mathcal{F}_T$ is proportional to D_T. We also denote by $P^{(1)}$ the extension[19] of the probability measures $P_T^{(1)}$ to the set of infinite paths.

Equation (5) shows that the long run behavior of $\frac{S_t}{D_t}$ is a key element in the determination of the asymptotic cumulative equity return. As seen in Proposition 5.1, this ratio is asymptotically given by $\left(\frac{S}{D}\right)_{I_K}$ and is constant. However, even though this convergence is an almost sure convergence under P, it is not clear whether or not the limit remains the same under $P^{(1)}$. Indeed, the restrictions of the measures P and $P^{(1)}$ on each sigma-algebra F_t are equivalent, but they are mutually singular on F_∞. Since D_t is a geometric Brownian motion with volatility σ, $W_t^{(1)} = W_t - \sigma t$ is a Brownian motion under $P^{(1)}$. Therefore, by the same argument as in (1)-(2), the strong law of large numbers for W_t implies that $P^{(1)}$ is supported on the set of paths of W_t that grow as σt when $t \to \infty$, whereas $W_t/t \to 0$ under P.

The optimal consumption of agent i can be rewritten as follows

$$c_{it} \;=\; e^{-\rho_i^{(1)} b_i t}\, M_t^{-b_i}\, (Z_{it}^{(1)})^{b_i}\, c_{i0}$$

where

$$\rho_i^{(1)} \;=\; \rho_i - \delta_i\sigma^2 \qquad \textit{and} \qquad Z_{it}^{(1)} \;=\; e^{\delta_i W_t^{(1)} - \frac{1}{2}\delta_i^2 t}$$

and where $W_t^{(1)}$ is a standard Brownian motion under $P^{(1)}$. Thus, under this new measure everything looks the same, apart from the fact that agents have discount rates given by $\rho_i^{(1)} \;=\; \rho_i - \delta_i\sigma^2$ and that the drift is given by $\mu^{(1)} = \mu + \sigma^2$. This means that, under $P^{(1)}$, the surviving agent is no more agent I_K but agent $A(1)$ characterized by

[19]The existence of such a probability measure is guaranted by the Kolmogorov extension Theorem.

$$(\rho^{(1)}_{A(1)} + \gamma_{A(1)}(\mu^{(1)} - \frac{1}{2}\sigma^2) + \frac{1}{2}\delta^2_{A(1)}) \ = \ \min_i (\rho^{(1)}_i + \gamma_i(\mu^{(1)} - \frac{1}{2}\sigma^2) + \frac{1}{2}\delta^2_i).$$

This suggest that survival and long run impact are different concepts. In the following we will illustrate the fact that the long run impact is determined by different agents depending on the asset under consideration.

Intuitively, one would expect from Equation (5) that the cumulative equity returns converge to those determined by agent $A(1)$. In fact, the long run return in the homogeneous economy populated by agent $A(1)$ only provides a lower bound for the long run return in our economy. Since a change of probability leads to a change of surviving agent, it is possible to obtain other lower bounds by the introduction of well chosen artificial probabilities. The next proposition provides such lower bounds based on the consideration of a parametrized family of such artificial probabilities.

Proposition 5.2 *Let $t = \lambda T$. We have almost surely under P :*

$$\lim\inf_{T\to\infty} (T-t)^{-1} \log R(t,T) \geq \mu + \max_\alpha \left(-\frac{1}{2}\sigma^2(1-\alpha)^2 + \left(\frac{S}{D}\right)^{-1}_{A(\alpha)} \right)$$

where $A(\alpha)$ is characterized by

$$\rho_{A(\alpha)} - \delta_{A(\alpha)}\sigma^2\alpha + \gamma_{A(\alpha)}(\mu - \frac{1}{2}\sigma^2 + \sigma^2\alpha) + \frac{1}{2}\delta^2_{A(\alpha)}$$
$$= \min_i \left(\rho_i - \delta_i\sigma^2\alpha + \gamma_i\left(\mu - \frac{1}{2}\sigma^2 + \sigma^2\alpha\right) + \frac{1}{2}\delta_i^2 \right). \quad (6)$$

Example 5.1 *Assume that all agents have the same level of risk aversion γ and the same time preference parameter ρ, but have heterogeneous beliefs that vary continuously taking values in $[\delta_{\min}, \delta_{\max}]$ with $\delta_{\min} < 0$ and $\delta_{\max} > [(\gamma-1)\sigma + 1]\,\sigma^2 > 0$. We have*

$$\rho - \delta_{A(\alpha)}\sigma^2\alpha + 0.5\delta^2_{A(\alpha)} = \min_i(\rho - \delta_i\sigma^2\alpha + \frac{1}{2}\delta_i^2)$$

which leads to

$$\delta_{A(\alpha)} = \sigma^2\alpha$$

as long as $\sigma^2\alpha \in [\delta_{\min}, \delta_{\max}]$. *We have then*

$$\lim\inf_{T\to\infty} (T-t)^{-1} \log\, R(t,T)$$

$$\geq \gamma\mu+\rho-0.5(\gamma-1)\sigma^2-(\gamma-1)^2\sigma^2+\sigma^2 \max_{\alpha\in\left[\frac{\delta_{\min}}{\sigma^2},\frac{\delta_{\max}}{\sigma^2}\right]} \left(-0.5(1-\alpha)^2+(\gamma-1)\sigma\alpha\right)$$

The maximum is reached for

$$\alpha^* = (\gamma-1)\sigma+1$$

which gives

$$\delta_{A^*} = \sigma^2((\gamma-1)\sigma+1) > 0.$$

By construction, the long run return in this economy is higher than the long run return in the economy populated by agent A^* *only. Note also that*

$$\mu + \left(\frac{S}{D}\right)_i^{-1} = \mu+\rho+(\gamma-1)(\mu-0.5\sigma^2+\sigma\delta_i+(1-\gamma)\sigma^2)$$

which means that the long run return in the homogeneous economies increases with δ_i *if and only if* $\gamma > 1$. *In this case we also have that the long run return in the homogeneous economy goes to infinity when* δ_i *goes to infinity. Consequently, for* $\gamma > 1$, *we have that the long run return is higher than the long run return in the homogeneous economy populated by agent* A^* *with* $\delta_{A^*} = \sigma^2((\gamma-1)\sigma+1) > 0$. *The long run return in this economy corresponds then to the long run return in a homogeneous economy populated by agent* B *with* $\delta_B \geq \delta_A > 0$ *and such that*

$$\lim\inf_{T\to\infty} (T-t)^{-1} \log\, R(t,T) = \mu+\rho+(\gamma-1)(\mu-0.5\sigma^2+\sigma\delta_B+(1-\gamma)\sigma^2).$$

As we show in Sections 6 and 7 below, in this economy we have that the long run return is determined by the agent with $\delta = \delta_B > 0$ *while the long run discount rate is determined by the agent with* $\delta = \delta_{\min} < 0$ *and the long run short rate, volatility and stock price are determined by the agent with* $\delta = 0$, *which is the only surviving agent.*

This example illustrates the fact that the agent who drives the long run discount rate may be different from the agent who drives the long run risky returns and both of them may be different from the surviving agent who drives the instantaneous risk free rate in the long run. Furthermore, the long run risk premium (the spread between the long run risky and riskless returns) is higher than the instantaneous risk premium. The presence of heterogeneity modifies the long term relation between risk and return leading to an additional premium in the long run.

5.3 Optimal Portfolios

For simplicity, everywhere in this section we assume that every agent i is endowed with a fixed number η_i of stock shares, so that we do not have to include the replicating portfolio for the agent's endowment.

Let us consider the investment strategy of agent i in the risky asset and in the riskless asset that permits to implement the equilibrium consumption process c_{it}. Such a strategy is characterized by a process π_{it} that corresponds to the amount of money held in the risky asset at date t by the agent under consideration. If we denote by w_{it} the financial wealth of agent i at date t corresponding to this strategy, we have

$$dw_{it} = w_{it}(r_t dt + \pi_{i\,t}(S_t^{-1}(dS_t + D_t dt) - r_t dt)) - c_{it}dt \tag{7}$$

$$= w_{it}(r_t\, dt + \pi_{i\,t}\sigma_t(\theta_t dt + dB_t)) - c_{it}dt. \tag{8}$$

In the following, we denote by π_{it}^{myopic} the myopic (instantaneously mean variance efficient) portfolio given by

$$\pi_{it}^{\text{myopic}} = \frac{\delta_i + \theta_t}{\gamma_i \sigma_t}$$

and we denote by $\pi_{it}^{\text{hedging}} = \pi_{it} - \pi_{it}^{\text{myopic}}$ the hedging component of the optimal portfolio, i.e. the component that hedges against future fluctuations of the market risk premium.

The following proposition characterizes the optimal portfolio and provides its long run composition.

Proposition 5.3 *1. The optimal portfolio is given by*

$$\sigma_t \pi_{it} = \theta_t + \frac{E_t\left[\int_t^\infty (b_i\delta_i + (b_i - 1)\theta_\tau) M_\tau c_{i\tau} d\tau\right]}{E_t\left[\int_t^\infty M_\tau c_{i\tau} d\tau\right]}$$

In particular,

$$\min_j \theta_j + \min_j (b_i\delta_i + (b_i - 1)\theta_j) \le \sigma_t \pi_{it} \le \max_j \theta_j + \max_j (b_i\delta_i + (b_i - 1)\theta_j)$$

2. *If we further assume that* $\gamma_i > 1$, *for all* i, *then, almost surely under* P,

$$\lim_{t\to\infty} \pi_{it} = \frac{\delta_i + \theta_{I_K}}{\sigma\gamma_i}.$$

3. *Suppose risk aversion is homogeneous. The sign of* $(1-b)\,\pi_{it}^{\text{hedging}}$

   - *is positive if the sequences*

$$b\sigma\delta_i + \frac{1}{2}b(1-b)\delta_i^2 + b\rho_i - 2b^2(\max_j \theta_j + \delta_i)\delta_i\,, \tag{9}$$

$$b\sigma\,\delta_i + \frac{1}{2}b(1-b)\delta_i^2 + b\rho_i - 2b^2(\min_j \theta_j + \delta_i)\delta_i \tag{10}$$

     are both anti-comonotone with (δ_i)*;*

   - *is negative if the sequences* (9)-(10) *are both comonotone with* (δ_i)*.*

The long run risky portfolio corresponds then, for each agent, to his optimal risky portfolio when facing an asset whose risk premium corresponds to the long run risk premium of our heterogeneous economy, that is to say the risk premium that would prevail in the economy populated by agent I_K only.

6 State price density, consumption shares and survival issues

In this section we first analyze how the state price density M fluctuates with W_t (or equivalently with aggregate endowment) as well as its long run behavior.

Proposition 6.1 • *For each state of the world, the state price density lies in the range bounded by the lowest and the highest individual state price densities*

$$\min_{1\le i\le N} M_i \;\le\; M \;\le\; \max_{1\le i\le N} M_i.$$

- *The long run behavior of the state price density is given by* $M_t \sim c_{I_K0}^{\gamma_{I_K}} M_{I_Kt}$.

- *If all the state price densities* $M_{it} = M_i(t, W_t)$ *are decreasing in* W_t*, then*

  - *the state price density* $M_t = M(t, W_t)$ *is also decreasing in* W_t*,*
  - *the state price density* $M_t = M(t, W_t)$ *satisfies*

$$\lim_{W_t \to -\infty} \frac{M(t, W_t)}{c_{\theta_{\max}0}^{\gamma_{I_{\theta_{\max}}}} M_{I_{\theta_{\max}}t}} = \lim_{W_t \to +\infty} \frac{M(t, W_t)}{c_{\theta_{\min}0}^{\gamma_{I_{\theta_{\min}}}} M_{I_{\theta_{\min}}t}} = 1$$

    almost surely under P.

When the agents have the same level of risk tolerance b (and possibly differ in their beliefs or in their time preference rates), it is easy to check that the equilibrium state price density is a weighted power average of the state price densities in homogeneous economies (the power being given by the common level of risk tolerance). In the general setting, the first point shows that the state price density M can still be interpreted as a kind of average of densities M_i. In the long run and in extreme states of the world, the state price density M is equivalent to the state price density that would prevail in an economy made of homogeneous agents with a different endowment distribution. This class of homogeneous agents is given by the agent who dominates the individual state price densities M_i in the considered states of the world: agent I_K asymptotically, agent $I_{\theta_{\max}}$ in the very bad states and agent $I_{\theta_{\min}}$ in the very good states.

The long run result implies, in particular, that except agent I_K, the agents have no *price impact* in the sense of Kogan et al. (2008, Definition 2) since we have for all $s > 0$,

$$\lim_{t \to \infty} \frac{M_{t+s}/M_t}{M_{I_K t+s}/M_{I_K t}} = 1.$$

However, we see in other sections that there may be price impact in the sense that the prices of assets may not be asymptotically the same as in the economy with only the agent who has the lowest survival index. Note however that this only holds for assets with very long finite maturities, such as zero coupon bonds.

Let us explore deeper these survival and dominance issues through an analysis of the behavior of the consumption shares and of the relative levels of risk tolerance.

As in Kogan et al. (2006) or Yan (2008), we say that investor i becomes extinct if $\lim_{t\to+\infty}\frac{c_{it}}{D_t}=0$, that she survives if extinction does not occur and that she dominates the market asymptotically if $\lim_{t\to+\infty}\frac{c_{it}}{D_t}=1$. We easily deduce from the properties of the individual state price densities obtained in Proposition 3.1 the following properties of the consumption shares and relative levels of risk tolerance, the first of which was obtained by Yan (2008).

Corollary 6.1

- *Only the agent with the lowest survival index survives and dominates the market asymptotically, i.e.,* $\lim_{t\to\infty}\frac{c_{it}}{D_t} = 0$ *for all* $i\neq I_K$*, and* $\lim_{t\to\infty}\frac{c_{I_K t}}{D_t} = 1$.
- *Only the agent with the lowest survival index impacts asymptotically the relative level of risk tolerance, i.e.,* $\lim_{t\to\infty}\omega_{it} = 0$ *for all* $i\neq I_K$*, and* $\lim_{t\to\infty}\omega_{I_K t} = 1$.
- *We have* $\lim_{W_t\to\infty}\omega_i(t,W_t) = \lim_{W_t\to\infty}\frac{c_i}{D}(t,W_t) = 0$ *for all* $i\neq I_{\theta_{\min}}$ *and* $\lim_{W_t\to\infty}\omega_{I_{\theta_{\min}}}(t,W_t) = \lim_{W_t\to\infty}\frac{c_{I_{\theta_{\min}}}}{D}(t,W_t)=1$*. We have* $\lim_{W_t\to-\infty}\omega_i(t,W_t) = \lim_{W_t\to-\infty}\frac{c_i}{D}(t,W_t) = 0$ *for all* $i\neq I_{\theta_{\max}}$ *and* $\lim_{W_t\to-\infty}\omega_{I_{\theta_{\max}}}(t,W_t) = \lim_{W_t\to-\infty}\frac{c_{I_{\theta_{\max}}}}{D}(t,W_t)=1$.
- *We have* $\frac{\partial\omega_i(t,W_t)}{\partial W_t} = \omega_{it}\left[b_i(\theta_t-\theta_i)-\sum_j\omega_{jt}b_j(\theta_t-\theta_j)\right]$ *and there is a shift following good news in the relative levels of risk tolerance towards agents with a relatively high* $b_i(\theta_t-\theta_i)$.

This implies that only the agent with the lowest survival index (resp. with the highest/lowest market price of risk) dominates the market in the sense of the consumption shares, or in the sense of the risk tolerance asymptotically (resp. in very bad/good states of the world). As previously seen, this agent is the agent who values the wealth more than the other agents in the considered state.

Note that the agent with the highest $b_i(\theta_t-\theta_i)$ is the most optimistic agent when there is only heterogeneity in beliefs, and is the least risk averse agent when there is only heterogeneity in risk aversion levels. In both cases this agent is the one who has the highest risk exposure and is then the most favored by good news.

7 Bond prices

The most striking result of this section is that each part of the (asymptotic) yield curve is dominated by different agents. We first start by considering the long run average discount rate.

As seen in Section 3, in the homogeneous economies the average discount rate between time t and T is the same for all (t, T) and given by the constant risk free rate. Indeed, we have in the homogeneous economy made of agent i only, $B_i(t, T) = e^{-(T-t)r_i}$, and $Y_i(t, T) = r_i$. The yield curves, representing, for all time t, the discount rates $Y_i(t, T)$ as a function of $T - t$, are the same for all time t and flat.

In the heterogeneous economy, the yield curves are not flat. The instantaneous discount rate defined by $\lim_{T \to t} Y(t, T)$ is given by the risk free rate $r_{t.}$ The next proposition characterizes the long run discount rate.

Proposition 7.1 *The long run average discount rate is determined by the agent with the highest savings motives, i.e., for all t,*

$$\lim_{T \to +\infty} Y(t, T) = r_{I_0}$$

almost surely under P.[20]

The same reasoning as above holds: when one agent dominates the individual price of an asset then she makes the price of that asset in the heterogeneous economy. As seen in Proposition 3.1, the agent with the highest savings motives dominates the price of the very long term bond because it is most attractive for her. That agent then drives the asymptotic average discount rate. This proposition is the extension, to the setting with three possible sources of heterogeneity (and many agents), of the proposition of Gollier and Zeckhauser (2005) for the case of heterogeneous time preference rates, of Wang (1996) for heterogeneous levels of risk aversion and of Jouini et al. (2010) for heterogeneous beliefs.

In the setting with heterogeneous time preference rates only, the same agent drives the long run discount rate and the long run risk free rate. Indeed, in that case, the agent with the lowest survival index is also the agent with the highest savings motives, namely the most patient agent. Apart from this

[20]Note that for a fixed finite t measures P and Q are equivalent, and so the convergence is also Q-almost surely.

setting, it is quite striking that the agent that drives the asymptotic average discount rate differs from the agent that drives the long run risk free rate, even though the discount rate is an average of the risk free rates. Indeed, we have $Y(t,T) = -\frac{1}{T-t}\log E_t^Q\left[\exp - \int_t^T r_s ds\right]$ where Q is the risk-neutral probability measure, with $r_s \to r_{I_K}$ while $Y(t,T) \to_{T\to\infty} r_{I_0}$. Analogously, we have $Y(t,T) = -\frac{1}{T-t}\log E_t\left[\frac{M_T}{M_t}\right]$ with $\frac{M_T}{M_t} \sim \frac{M_{I_K T}}{M_{I_K t}}$ while $Y(t,T) \sim Y_{I_0}(t,T)$.

In the case with heterogeneous beliefs only, the risk free rate converges to the rate of the most rational agent whereas the long run discount rate is driven by the most pessimistic agent.

In particular, Proposition 7.1 as well as Corollary 4.1 imply that when t is large enough, the yield curve representing $Y(t,T)$ as a function of $(T-t)$ is driven by the risk free rate of the agent with the lowest survival index (agent I_K) at one end of the yield curve, i.e., for small values of $(T-t)$, whereas at the other end, i.e., for $(T-t)$ large enough, it is driven by the risk free rate of the agent with the highest savings motives or equivalently the lowest risk free rate (agent I_0). The aim of the remainder of this section is to show that the yield curve is defined stepwise, and that each subinterval is associated with a given agent in the sense that the marginal rate on that subinterval corresponds to the rate in the economy made of that agent only. Moreover, that agent is the agent who most values a given zero coupon bond associated to the subinterval and is characterized by a maximization program involving a weighted average of the savings motives and of the survival index.

In order to show this, we first identify the relevant subintervals, as follows. In the homogeneous economy made of agent i only, the price, seen from date 0, of a zero coupon bond between time t and time T and in state ω is given by $E_t[M_{iT}] = e^{-r_i(T-t)}e^{-\kappa_i t\ -\theta_i W_t}$. This implies that for $\lambda \in [0,1]$, we have $E_{\lambda T}[M_{iT}] = e^{-l_i(\lambda)T}e^{-\theta_i W_{\lambda T}}$ where

$$l_i(\lambda) = [\lambda\kappa_i + (1-\lambda)r_i] = \left[\kappa_i - (1-\lambda)\frac{1}{2}\theta_i^2\right]$$

is a weighted average of the survival index and of the risk free rate. Since $(l_i(\lambda), \lambda \in [0,1])$ is a family of line segments, there exist pairs of values $((I_j, \lambda_j), j = 1, \cdots, K)$ such that

$$\min_i l_i(\lambda) = l_{I_j}(\lambda) \text{ for all } \lambda \in (\lambda_j, \lambda_{j+1})$$

where $\lambda_0 = 0$ and $\lambda_{K+1} = 1$. For example, for λ near 0, agent I_0 satisfies $r_{I_0} = \inf_i r_i$ and for λ near 1, agent I_K satisfies $\kappa_{I_K} = \inf_i \kappa_i$.

Intervals $(\lambda_j, \lambda_{j+1})$ are exactly those that will determine the stepwise behavior of the yield curve. This is basically due to the following: The index $l_i(\lambda)$ drives the asymptotic behavior of the price $E_{\lambda T}[M_{iT}]$ in the sense that

$$\lim_{T\to\infty} \frac{E_{\lambda T}[M_{iT}]}{E_{\lambda T}\left[M_{I_j T}\right]} = 0 \text{ for all } i \neq I_j \text{ when } \lambda \in (\lambda_j, \lambda_{j+1}) \tag{11}$$

This is due to the fact that the price $E_t[M_{iT}]$ involves both the state price density M_{it} whose long run behavior is driven by the survival index and the bond price $B_i(t,T)$, whose long run behavior is driven by the savings motives. For $\lambda = 0$, we retrieve the fact that agent I_0 (with the lowest risk free rate) dominates the prices of the zero coupon bond $B_i(0,T)$ when T is large enough. For $\lambda = 1$, we retrieve the fact that agent I_K (with the lowest survival index) dominates the state price densities M_{iT} for T large enough. For $\lambda \in (0,1)$, we obtain that agent I_j (with the lowest index $l_i(\lambda)$, mixing the survival index and the savings motives) dominates the prices $E_{\lambda T}[M_{iT}]$ when T is large enough.

In order to interpret how the agents dominating each subinterval are chosen, consider, for example, the case with heterogeneity in beliefs only. Agent I_0 is then the most pessimistic agent and agent I_K is the most rational agent. Agent I_1 is the most pessimistic agent once agent I_0 is excluded, agent I_2 is the most pessimistic agent once agents I_0 and I_1 are excluded, etc. Moreover, the intervals $(\lambda_j, \lambda_{j+1})$ on which $l_{I_j}(\lambda) = \min_i l_i(\lambda)$ are given by $\lambda_j = \frac{2\gamma\sigma}{2\gamma\sigma - \left(\delta_{I_{j-1}} + \delta_{I_j}\right)}$. Note that apart from agent I_K (who might be optimistic or pessimistic) all the agents I_j (for $j = 0, ..., K-1$) are pessimistic. This is due to the following: In the case with heterogeneity on the beliefs only, minimizing $\mathrm{l}_i(\lambda)$ amounts to minimizing the average of the survival index and of the risk free rate associated to the i-th agent. The survival index reaches its minimum for the lowest δ_i in absolute value (i.e., for the most rational agent), while the risk free rate increases with δ_i. Starting from the most rational agent, it is clear that the only way to possibly decrease $l_i(\lambda)$ consists in moving in the direction of more pessimism.

We are now in a position to state our main result on the bond prices and the yield curve.

Proposition 7.2 • *The bond prices satisfy*

$$E_t\left[M_{\alpha t}\right] \sim c_{I_j0}^{\gamma_{I_j}} E_t\left[M_{I_j\alpha t}\right] \text{ and } B\left(t,\alpha t\right) \sim \frac{c_{I_j0}^{\gamma_{I_j}}}{c_{I_K0}^{\gamma_K}} \frac{E_t\left[M_{I_j\alpha t}\right]}{M_{I_K t}}$$

for all $\alpha \in (\frac{1}{\lambda_{j+1}}, \frac{1}{\lambda_j})$ *almost surely under P.*

- *We have, for* $\alpha \in (\frac{1}{\lambda_j}, \frac{1}{\lambda_{j-1}})$,

$$Y\left(\alpha\right) \equiv \lim_{t\to\infty} Y(t,\alpha t) = \frac{1}{\alpha - 1}\left[\kappa_{I_K} - \alpha\, l_{I_j}\left(1/\alpha\right)\right]$$

almost surely under P.

and the convergence is uniform on compact subsets of $(1,\infty)$. *We have* $\lim_{\alpha\to 1} Y(\alpha) = r_{I_K}$ *and* $\lim_{\alpha\to\infty} Y(\alpha) = r_{I_0}$.

- *The marginal rates associated to the long run yield curve (the instantaneous forward rates) are given by*

$$\frac{d}{d\alpha}\left[Y(\alpha)(\alpha-1)\right] = r_{I_j}$$

on $(\frac{1}{\lambda_{j+1}}, \frac{1}{\lambda_j})$.

The above result provides then the shape of the long run yield curve. However, it is important to notice that, asymptotically, yield curves at different dates are obtained through homothetic transformations and not through translations. In other words, for t large enough, all yield curves will have the same shape, but at different scales.

Different segments of the (asymptotic) yield curve are determined by different agents with different characteristics. More precisely, the marginal discount rate for the interval $(\frac{1}{\lambda_{j+1}}, \frac{1}{\lambda_j})$ is determined by agent I_j.

Intuitively, we can interpret the individual agent interest rate r_i as the effective agent i's discount rate. The agent with the lowest discount rate is effectively the most patient and therefore determines the long end of the yield curve. On the other hand, the short end of the yield curve is determined by the single surviving agent. As the maturity changes from the short to the long end, the corresponding interest rate changes along the yield curve, switching sequentially between different agent's interest rates. Thus, for each particular agent i, Proposition 7.2 provides an explicit expression for the range of maturities that agent i's interest rate corresponds to. When there are only two agents in the economy, the short (long) end of the yield curve is determined by the agent with a higher (lower) individual interest rate. However, this is not anymore true when there are more than two agents in the economy because the single surviving agent K is not necessarily the agent with the lowest individual interest rate.

It is interesting to note that even though only one agent survives in the long term, non-surviving agents might continue to have an impact on the yield curve. One may argue that the impact of agent I_j is only between $(\frac{1}{\lambda_{j+1}}t, \frac{1}{\lambda_j}t)$ and is then at more and more distant horizons when t increases. However, we can construct examples where non surviving agents impact prices and where this impact does not vanish asymptotically, as illustrated in the following.

Example 7.1 *Assuming heterogeneity in beliefs only, we know that r_{I_K} corresponds to the risk free rate in the economy populated by the most rational agent and r_{I_0} corresponds to the risk free rate in the economy populated by the most pessimistic agent only. Let us consider an asset (a growing perpetuity) with a deterministic dividend flow $d_t = d_0 \exp(\hat{r}t)$ with $r_{I_0} < \hat{r} < r_{I_K}$. The price at date t of this asset in the economy populated by agent I_K only is given by*

$$p_t = d_0 \exp\left(r_{I_K} t\right) \int_t^{\infty} \exp\left((\hat{r} - r_{I_K})s\right) ds = \frac{d_0}{r_{I_K} - \hat{r}} \exp\left(\hat{r}t\right)$$

in terms of date t prices. On the other hand, the price p'_t of this asset in the heterogeneous economy is infinite in terms of date t prices. Indeed, if we denote by $\bar{r}_s$ the marginal discount rate (from date t point of view) at date s (i.e. $\bar{r}_s = -\frac{1}{B(t,s)}\frac{\partial B(t,s)}{\partial s}$) we know that $\bar{r}_s$ is arbitrarily close to r_{I_0} for s sufficiently large. More precisely, let s be such that $\bar{r}_v \leq \hat{r} - \varepsilon$ for $\varepsilon > 0$ and for all $v \geq s$. We have

$$\begin{aligned} p'_t &= d_0 \exp\left(\hat{r}t\right) \int_t^{\infty} \exp\left(\int_t^u (\hat{r} - \bar{r}_v) dv\right) du \\ &\geq d_0 \exp\left(\hat{r}t\right) \exp\left(\int_t^s (\hat{r} - \bar{r}_v) dv\right) \int_s^{\infty} \exp\left(\int_s^u (\hat{r} - \bar{r}_v) dv\right) du \end{aligned}$$

and it is easy to see that the last integral is infinite and so is p'_t.

We will now state the last result of this section that provides an intuitive link between survival and the long run bond price impact of Proposition 7.2. In order to state it, we need some definitions.

Recall that the T-forward measure Q^T is defined by

$$dQ^T \;=\; \frac{e^{-\int_0^T r_s ds}}{\beta(0\,,\,T)}\,dQ\,.$$

Consequently,

$$E_t^{Q^T}[X] \;=\; \frac{E_t[M_T\,X]}{E_t[M_T]}$$

for any random variable X.

We will say that an agent i survives with respect to the family of measures $\{Q^T\}$ for $t\,=\,\lambda\,T$ if

$$\limsup_{T\to\infty}\;E_{\lambda T}^{Q^T}[c_{i\,T}\,D_T^{-1}]\;>\;0 \tag{12}$$

with positive P-probability. We have the following result.

Proposition 7.3 *An agent i survives with respect to the family of T-forward measures for $t=\lambda T$ if and only if*

$$l_i(\lambda)\;=\;\min_j\,l_j(\lambda)\,.$$

Consequently, agent i has an impact on the bond price $B(\lambda T, T)$ if and only he survives with respect to Q^T.

The result of Proposition 7.3 is very intuitive. As we mention above, the segment of the yield curve determined by agent i corresponds to the maturities for which the long-run discount rate coincides with the individual rate of agent i. Survival with respect to the family of T-forward measures precisely means that the effective discount rate of the agent corresponds to the maturity of the forward measure. It is also interesting to note that the market price of risk θ_t^T under the T-forward measure coincides with the T-forward expectation of the true market price of risk,

$$\theta_t^T\;=\;E_t^{Q^T}[\theta_T]\,. \tag{13}$$

In particular, it follows from (12) that

$$\lim_{T\to\infty}\theta_{\lambda T}^T\;=\;\lim_{T\to\infty}E_{\lambda T}^{Q^T}[\theta_T]\;=\;\theta_{I(j)}$$

for $j\in(\frac{1}{\lambda_{j+1}},\frac{1}{\lambda_j})$ almost surely under P.

7.1 Examples

Example Suppose that there are two agents with parameters $(\gamma_1, \rho_1, \delta_1) = (5, 0.95, 0.5)$ and $(\gamma_2, \rho_2, \delta_2) = (2.5, 0.98, -1)$. That is, agent 2 is less risk averse, more impatient and pessimistic. In this case, a direct calculation shows that agent 1 is the single surviving agent with $r_1 \approx 1.1$ whereas $r_2 = 0.93 < r_1$ and hence agent 2 will dominate the upper end of the yield curve. Our asymptotic results predict that the yield curve $Y(t, \tau)$ should be close to r_1 in the short end, and close to r_2 in the long end. Figure 1 below illustrates how the yield curve evolves with the natural state variable, the consumption ratio c_{1t}/c_{2t}. Clearly, the whole yield curve should become flat at the level r_1 (respectively, r_2) when consumption ratio is sufficiently high (respectively, sufficiently low). We see that this is indeed true, but the process is much slower for the short end than for the long end of the curve. Namely, the yield curve gets almost flat and close to r_2 for maturities above 30 years already when c_{10}/c_{20} is less than 0.25. By contrast, when c_{10}/c_{20} equals 4, the yield curve shows absolutely no signs of convergence to its long run value of r_1, and even for $c_{10}/c_{20} = 100$ it starts significantly deviating from r_1 for long maturities, approaching r_2. Interestingly enough, for moderate values of the consumption ratio c_{10}/c_{20}, the short end of the yield curve is strictly above the maximal individual rate r_1.

In view of Proposition 7.3, it is instructive to understand the relationship between survival and price impact in this example economy. To this end, we provide the plot of the drift of $\log(c_{1t}/c_{2t})$ under different measures as a function of c_{1t}/c_{2t} in Figure 2 below. These drifts are computed in Appendix B.[21]

As we can see from this figure, the drift of the ratio under the physical measure P is essentially flat and always positive. This stands in perfect agreement with the fact that only agent 1 survives under P. Thus, on average, the quotient c_{1t}/c_{2t} will be always growing exponentially fast even for very low levels of c_{1t}/c_{2t}. The behavior is drastically different under the risk neutral measure Q and the T-forward measure Q^T with $T = 50$. By Proposition 7.3, agent 2 should be the only one surviving under Q^T when T is large, and so we expect the drift of $\log(c_{1t}/c_{2t})$ to be negative when c_{1t}/c_{2t} is not too large. This theoretical prediction is in perfect agreement with Figure 2. Indeed, the drift is negative for $c_{1t}/c_{2t} < 17$. However, the drift exhibits an unexpected pattern and is first decreasing in c_{1t}/c_{2t}, and only then starts increasing and

[21]We thank the anonymous referee for suggesting us to make this very intuitive plot.

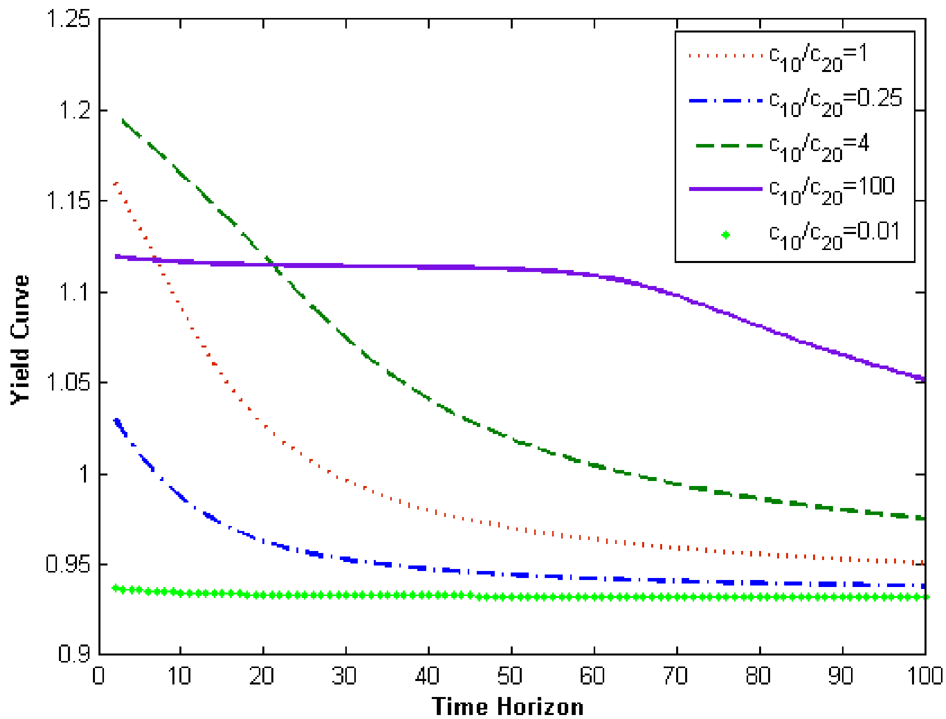

Figure 1: Yield curve for different values of the consumption ratio c_1/c_2

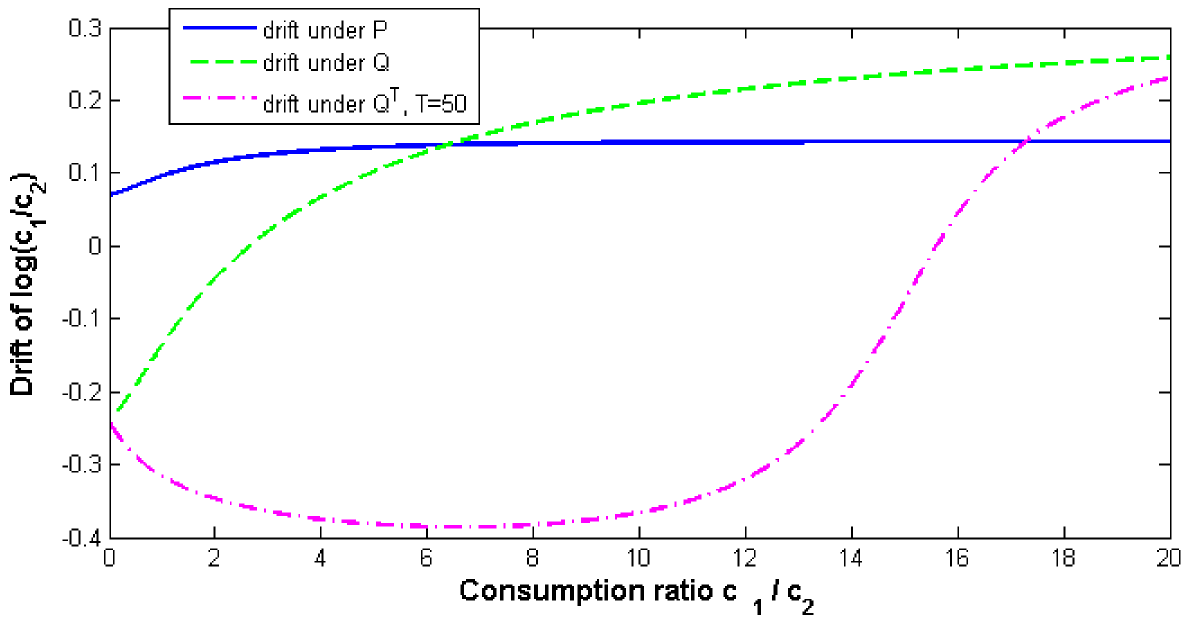

Figure 2: The drift of the log consumption ratio $\log(c_{1t}/c_{2t})$ as a function of the consumption ratio c_{1t}/c_{2t}.

gets positive. The reason is that, naturally, the difference between the drift under P and the drift under Q^T is determined by the market price of risk θ_t^T given by (13). Since agent 2 is quite pessimistic, we have $\theta_2 < 0 < \theta_1$. For moderate values of c_{1t}/c_{2t}, agent 2 dominates the economy and we have $\theta_t^T \approx \theta_2 < 0$, whereas $\theta_t^T \approx \theta_1$ for large values of c_{1t}/c_{2t}. Therefore, θ_t^T changes sign when c_{1t}/c_{2t} increases and starts converging to θ_1, pushing the drift up.

The behavior of the drift under the risk neutral measure is different: it is always increasing in c_{1t}/c_{2t}, but, as the drift under Q^T, it is negative for small values of c_{1t}/c_{2t}. The behavior of the drift under Q is different from that under Q^T because the market price of risk θ_t increases linearly from the value θ_2 at $c_{1t}/c_{2t} = 0$ to the value θ_1 for large c_{1t}/c_{2t}, whereas θ_t^T stays approximately equal to θ_2 for moderate values of c_{1t}/c_{2t}, and only then starts converging to θ_1. Note finally that, as in (1)-(2), the measures P and Q are equivalent when restricted to $\mathcal{F}_t$, but they are mutually singular for $t = \infty$. Indeed, since $\theta_t \to \theta_1$ a.s. under P, the measure Q is supported on the paths of W_t such that $W_t/t \sim \theta_1$ as $t \to \infty$.

Note that Figure 2 suggests that both agents 1 and 2 may happen to survive under Q. Quite remarkably, this is indeed true and it is possible to explicitly characterize the long run behavior of the economy under Q, as is shown by the following proposition.

Define

$$\begin{aligned}
\zeta_1 &= \gamma_2(b_2 - b_1)\theta_1^2 + \gamma_2(b_2\delta_2 - b_1\delta_1)\theta_1 + \kappa_2 - \kappa_1 \\
\zeta_2 &= \gamma_1(b_2 - b_1)\theta_2^2 + \gamma_1(b_2\delta_2 - b_1\delta_1)\theta_2 + \kappa_2 - \kappa_1
\end{aligned}$$

A direct (but tedious) calculation shows that

$$\zeta_1 - \zeta_2 = (\theta_1 - \theta_2)^2 > 0\,.$$

Proposition 7.4 *Let $n = 2$. The following is true:*

- *If $\zeta_1 > \zeta_2 > 0$ then only agent 1 survives in the long run under Q, that is $c_{1t}/c_{2t} \to +\infty$ almost surely under Q;*
- *If $0 > \zeta_1 > \zeta_2$ then only agent 2 survives in the long run under Q, that is $c_{1t}/c_{2t} \to 0$ almost surely under Q;*
- *If $\zeta_1 > 0 > \zeta_2$ then, for Q almost every path, we have either $\lim_{T\to\infty} c_{1T}/c_{2T} =$*

$+\infty$ *or* $\lim_{T\to\infty} c_{1T}/c_{2T} = 0$ *and*

$$
\begin{aligned}
Q\left[\lim_{T\to\infty} c_{1T}/c_{2T} = +\infty | \mathcal{F}_t\right] &= \phi(c_{1t}/c_{2t}) \\
Q\left[\lim_{T\to\infty} c_{1T}/c_{2T} = 0 | \mathcal{F}_t\right] &= 1 - \phi(c_{1t}/c_{2t})
\end{aligned}
\tag{14}
$$

where

$$
\phi(x) = \frac{\int_{-\infty}^{x} \exp\left(-2\int_0^{\xi} \frac{b(y)dy}{\sigma^2(y)}\right) d\xi}{\int_{-\infty}^{+\infty} \exp\left(-2\int_0^{\xi} \frac{b(y)dy}{\sigma^2(y)}\right) d\xi}
$$

with

$$
b(y) = b_2\kappa_2 - b_1\kappa_1 + (b_1 - b_2)r(y) - 0.5(b_1 - b_2)\theta^2(y) - \theta(y)(b_1\delta_1 - b_2\delta_2)
$$

and

$$
\sigma(y) = (b_1 - b_2)\theta(y) + (b_1\delta_1 - b_2\delta_2)
$$

and

$$
\theta(y) = \sum_{i=1}^{2} \theta_i\, \omega_i(y)\ ,\ r(y) = \sum_{i=1}^{2} \omega_i(y) r_i + \frac{1}{2}\sum_{i=1}^{2} (1 - b_i)\,(\theta_i - \theta(y))^2 \omega_i(y)
$$

with

$$
\omega_1(y) = \frac{b_1 e^y}{b_1 e^y + b_2}\ ,\ \omega_2(y) = 1 - \omega_1(y)\,.
$$

The result of Proposition 7.4 is quite remarkable: it shows that, for an open set of parameters, it is possible that **both** agents survive in the long run with positive probability, even though they never survive simultaneously. Furthermore, the corresponding state prices (14) depend on the consumption allocation (c_{1t}, c_{2t}). This fact may have very important economic consequences, as is illustrated by the following example: consider the price $F_{t,T}$ of a futures contract whose payoff is some function $f(r_T)$. of the short rate r_T at maturity. Then,

$$
F_{t,T} = E_t^Q[f(r_T)]\,.
$$

When $T \to \infty$, formula (14) implies

$$
F_{t,T} \to \phi(c_{1t}/c_{2t}) f(r_1) + (1 - \phi(c_{1t}/c_{2t})) f(r_2)\,.
$$

In particular, the price of a futures contract $F_{0,T}$ with even a very long maturity will depend on the initial consumption allocation, which in turn depends in a very non-trivial way on the initial endowments of the agents. Since, generally speaking, equilibrium allocations may be non-unique, this also means that the long-run long run behavior may also be non-unique. To the best of our knowledge, this is a new phenomenon that has never been shown for this class of models before.

Example. In the case where only ρ varies, i.e., $U = [\rho_{\min}, \rho_{\max}] \times \{\gamma\} \times \{\delta\}$, we have

$$(\rho(\lambda), \gamma(\lambda), \delta(\lambda)) = (\rho_{\min}, \gamma, \delta)$$

and the long run term structure is constant. The whole long run yield curve is associated to the lowest level of impatience.

Example. Consider now the case where only γ varies. More precisely, suppose $U = \{\rho\} \times [\gamma_{\min}, \gamma_{\max}] \times \{\delta\}$. It is shown in Appendix that for the case where the economy is shrinking, $\mu < \sigma^2/2$, the whole yield curve (which is flat in this case) is associated to a single agent (the most risk-averse agent with $\gamma = \gamma_{\max}$, or the least risk-averse agent with $\gamma = \gamma_{\min}$, depending on how large is $\gamma_{\max}$). When the economy is growing, if the highest risk aversion is large enough, the yield curve is determined for short horizons by the agent with the lowest level of risk aversion (i.e., $\gamma = \gamma_{\min}$) and for long horizons by the agent with the highest level of risk aversion (i.e., $\gamma = \gamma_{\max}$). We have then two different habitats and the more distant one in time is associated to a higher level of risk aversion than the less distant one. As noted by Wang (1996), long term bonds are more attractive to more risk averse agents as hedging instruments against future downturns of the economy. Indeed, the more risk averse investors are more averse to low levels of future consumption. Consequently they exert a stronger influence on their equilibrium price. However, $\gamma_{\max}$ should be large enough with respect to $\gamma_{\min}$, for this phenomenon to occur. If not, we may have an inversion : $\gamma_{\max}$ determines the short term rates and $\gamma_{\min}$ the long term ones.

8 Conclusions

We study equilibrium in a complete financial market, populated by CRRA agents who differ in risk aversion, in beliefs on the growth of the economy, and in time preference rates. We show that the market price of risk is a risk tolerance-weighted average of the market prices of risk that would arise in the agent-corresponding homogeneous economies, while this is not true for the risk-free rate. In the long run, as time increases, these two quantities converge to those of the homogeneous economy corresponding to the surviving agent, which is the one with the lowest survival index. We obtain analogous results for the price-dividend ratio and price volatility. On the other hand, we construct examples in which the (asymptotic) price of an asset is not necessarily the one that would arise in the homogeneous economy corresponding to the surviving agent, thus illustrating the long-term impact of non-surviving agents. In our model the market price of risk is always decreasing in the level of the aggregate endowment, in agreement with empirical observations. The average discount rate and the long run risk free rate may differ from each other, and we show, more generally, that each part of the (asymptotic) yield curve is dominated by different agents. Moreover, the two agents driving the short end and the long end of the yield curve may differ from the agent who drives the long run risky returns. Furthermore, the long run risk premium may be higher than the instantaneous risk premium. Additionally, we obtain results on the agents' optimal portfolios, and results on long run behavior of the state price density, agents' consumption shares and agents' risk tolerances, both as time increases and as aggregate wealth takes extreme values. In future work, it would be of interest to see how our results extend to economies with non-CRRA agents, with more general aggregate dividend process, and with agents who update their beliefs on the growth of economy.

A Appendix

The following lemma provides an expression of the state price density M as well as bounds on M in terms of the state price densities in homogeneous economies. It is a direct analog of Lemma A-1 from Cvitanić and Malamud (2010).

Lemma A.1 *1. Letting $F\,(a_1, ..., a_n)$ be the function defined as the unique solution to $\sum_{i=1}^{N} F^{-b_i} a_i^{b_i} = 1$, we have $M = F\,(c_{10}^{\gamma_1} M_1, ..., c_{N0}^{\gamma_N} M_N)\,.$*

2. Let $\Gamma \geq 1$ be such that $\Gamma\, b_i > 1$ for all i and $\gamma \leq 1$ be such that $\gamma\, b_i \leq 1$

for all i . Then,

$$\left(\sum_{i=1}^{N} c_{i0}^{\gamma_i/\gamma} M_i^{1/\gamma}\right)^{\gamma} \leq M \leq \left(\sum_{i=1}^{N} c_{i0}^{\gamma_i/\Gamma} M_i^{1/\Gamma}\right)^{\Gamma}. \tag{15}$$

Proof of Lemma A.1

In the setting with heterogeneous agents, the first-order conditions of the agents optimization programs (O_{iM}) give us the existence of Lagrange multipliers λ_i such that, for all i, the equilibrium state price density satisfies

$$M_t = \lambda_i e^{-\rho_i t} Z_{it} c_{it}^{-\gamma_i}. \tag{16}$$

Since prices are in terms of date 0 consumption goods, we have $M_0 = 1$ and $\lambda_i = c_{i0}^{\gamma_i}$.

Let

$$b_i \;=\; 1/\gamma_i$$

denote the relative risk tolerance. The optimal consumption of investor i is given by

$$c_{it} = e^{-\rho_i b_i t} M_t^{-b_i} Z_{it}^{b_i} c_{i0} = \left(c_{i0}^{\gamma_i} M_{it}\right)^{b_i} D_t M_t^{-b_i}.$$

In equilibrium we require that

$$\sum_{i=1}^{n} c_{it} = D_t$$

or equivalently

$$\sum_i \left(c_{i0}^{\gamma_i} M_{it}\right)^{b_i} M_t^{-b_i} = 1. \tag{17}$$

Let $F(a_1, \cdots, a_n)$ be the function defined as the unique solution to

$$\sum_i F^{-b_i} a_i^{b_i} = 1. \tag{18}$$

Then, a direct consequence of the equilibrium equation is that

$$M = F(c_{10}^{\gamma_1} M_1, \cdots, c_{N0}^{\gamma_1} M_N).$$

Let $\Gamma \geq 1$ be such that $\Gamma b_i > 1$ for all i and $\gamma \leq 1$ be such that $\gamma\, b_i \leq 1$ for all i . We have $\sum_i \left(c_{i0}^{\gamma_i} M_{it}\right)^{b_i} M_t^{-b_i} = 1$ which gives $\left(c_{i0}^{\gamma_i} M_{it}\right) M_t^{-1} \leq 1$ and

$$\sum_i \left(c_{i0}^{\gamma_i} M_{it}\right)^{\frac{1}{\Gamma}} M_t^{-\frac{1}{\Gamma}} \geq 1 \geq \sum_i \left(c_{i0}^{\gamma_i} M_{it}\right)^{\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}}.$$

The bounds on M_t follow from there.

B Proofs for Section 4

Proof of Proposition 4.1

Assume that M can be written in the representation

$$dM_t = M_t(-r_t dt - \theta_t dW_t).$$

The risk free rate is then given by r_t and the market price of risk is given by θ_t. Since

$$M_t = F(c_{10}^{\gamma_1} M_{1t}, \cdots, c_{N0}^{\gamma_N} M_{Nt})$$

we have,

$$dM_t = \sum_i c_{i0}^{\gamma_i} F_{a_i} dM_{it} + \frac{1}{2} \sum_{i\,,\,j} c_{i0}^{\gamma_i} c_{j0}^{\gamma_j} F_{a_i a_j} d\langle M_{it}, M_{jt}\rangle. \tag{19}$$

By definition,

$$dM_{it} = M_{it} \left(-r_i dt - \theta_i dW_t\right) \tag{20}$$

and the formula

$$\theta_t = \sum_{i=1}^{N} \omega_{it} \theta_i,$$

follows directly. As far as the risk free rate is concerned, we have

$$d\langle M_{it}, M_{jt}\rangle = M_{it} M_{jt} \theta_i \theta_j dt. \tag{21}$$

which with (52) leads to

$$\begin{aligned} r_t = & \sum_{i=1}^{N} \omega_{it} r_i - \left(\sum_{i=1}^{N} \theta_i \omega_{it}\right) \left(\sum_{j=1}^{N} \theta_j \left(1 - b_j\right) \omega_{jt}\right) \\ & + \frac{1}{2} \left(\sum_{k=1}^{N} \left(1 - b_k\right) \omega_{kt}\right) \left(\sum_{i=1}^{N} \theta_i \omega_{it}\right)^2 + \frac{1}{2} \sum_{i=1}^{N} \left(1 - b_i\right) \theta_i^2 \omega_{it}. \end{aligned}$$

Proof of Corollary 4.1

The first point is immediate.

Using the inequality

$$a^2 + b^2 \geq 2ab,$$

we get

$$\frac{1}{2}\left(\sum_k |1-b_k|\,\omega_{kt}\right)\left(\sum_i \theta_i\omega_{it}\right)^2 + \frac{1}{2}\sum_i |1-b_i|\,\theta_i^2\omega_{it}$$
$$\geq \left(\sum_k |1-b_k|\,\omega_{kt}\right)^{1/2}\left(\sum_i \theta_i\omega_{it}\right)\left(\sum_i |1-b_i|\,\theta_i^2\omega_{it}\right)^{1/2}. \quad (22)$$

Now, applying the Cauchy-Schwarz inequality

$$\left(\sum_i x_i^2\right)^{1/2}\left(\sum_i y_i^2\right)^{1/2} \geq \sum_i x_i y_i$$

to

$$x_i = |1-b_i|^{1/2}\,\theta_i\omega_{it}^{1/2} \text{ and } y_i = |1-b_i|^{1/2}\,\omega_{it}^{1/2}$$

we get the first or the second inequality depending on the sign of the $(1-b_i)$ s. The limits are obtained from the behavior of the ω_i s in extreme states of the world.

The long run results are immediate from Corollary 6.1.

Proof of Corollary 4.2

We have,

$$\omega_{it} = c_{i0}^{\gamma_i} M_{it} F^{-1} F_{a_i}$$

and therefore

$$\frac{d\omega_{it}(W_t)}{dW_t}$$
$$= c_{i0}^{\gamma_i}\left(\frac{dM_{it}}{dW_t}F^{-1}F_{a_i} - M_{it}F^{-2}F_{a_i}\sum_j c_{j0}^{\gamma_j}F_{a_j}\frac{dM_{jt}}{dW_t} + M_{it}F^{-1}\sum_j c_{j0}^{\gamma_j}F_{a_i a_j}\frac{dM_{jt}}{dW_t}\right)$$
$$= -\omega_{it}b_i\theta_i + \omega_{it}\theta_t - \omega_{it}\sum_j \omega_{jt}\,(1-b_j)\theta_j - \omega_{it}(1-b_i)\theta_t + \omega_{it}\theta_t\sum_k (1-b_k)\,\omega_{kt}\,. \quad (23)$$

Therefore,

$$\begin{aligned}\frac{d\theta_t}{dW_t} &= \sum_i \theta_i \frac{d\omega_{it}(W_t)}{dW_t} \\ &= \sum_i \omega_{it}\, b_i\, \tilde{\gamma}_i \,(\gamma_i\sigma - \theta_i \,+\, \theta_t \,-\, \theta_t \sum_i \omega_{i\,t}\, b_i\,(\gamma_i\sigma - \theta_i + \theta_t) \\ &= -\sum_i \omega_{it}\, b_i\, \theta_i^2 \,+\, 2\,\theta_t \sum_i \omega_{it}\, b_i\, \theta_i \,-\, \theta_t^2 \sum_i \omega_{it}\, b_i \;.\end{aligned}$$

Applying the inequality $a^2 + b^2 \geq 2|ab|$ to

$$a^2 = \sum_i \omega_{it}\, b_i\, \theta_i^2 \;,\; b^2 = \theta_t^2 \sum_i \omega_{it}\, b_i$$

together with the Cauchy–Schwarz inequality

$$\left(\sum_i \omega_{it}\, b_i\, \theta_i^2 \sum_i \omega_{it}\, b_i\right)^{1/2} \geq \sum_i \omega_{it}\, b_i\, |\theta_i\,|$$

which is what had to be proved

Proof of Corollary 4.3

$$\frac{dr_t}{dW_t} = \frac{d}{dW_t}\Big[\sum_i \omega_i r_i - \theta_t \left(\sum_j \theta_j(1-b_j)\omega_{jt}\right) + \frac{1}{2}\left(\sum_k (1-b_k)\omega_{kt}\right)\theta_t^2 + \frac{1}{2}\sum_i (1-b_i)\theta_i^2 \omega_{i\,t}\Big] \quad (24)$$

Given $X = (x_1, \ldots, x_N)$, denote

$$\mathcal{E}(x) = \sum_i \omega_{it} x_i$$

Since the weights sum up to one, this is an expectation on $\{1, \cdots, n\}$ and we can also define Cov_ω and Var_ω. Let

$$R = (r_i)_{i=1,\cdots,N}, \qquad \beta = (b_i)_{i=1,\cdots,N} \text{ and } \Theta = (\theta_i)_{i=1,\cdots,N}.$$

Then, a direct but tedious calculation implies that

$$\frac{dr_t}{dW_t} = -\mathrm{Cov}_\omega(\beta\Theta\,,\ R)\ +\ \mathcal{E}(\Theta)\,\mathrm{Cov}_\omega(\beta, R)$$

$$+\,\mathcal{E}(\Theta)\mathrm{Cov}_\omega(\beta\Theta,\Theta) - \mathcal{E}(\Theta)\mathrm{Var}(\beta\Theta) - \mathcal{E}(\Theta)^2\mathrm{Cov}_\omega(\Theta,\beta) + \mathcal{E}(\Theta)^2\mathrm{Cov}_\omega(\Theta\beta,\beta)$$

$$+\,\frac{1}{2}\mathcal{E}(\Theta)^2\Big(\mathrm{Cov}_\omega(\Theta\beta,\beta) - \mathcal{E}(\Theta)\,\mathrm{Var}_\omega(\beta)\Big)$$

$$+\ \frac{1}{2}\Big(\mathrm{Cov}_\omega(\beta\,(\beta-1)\Theta^2,\Theta) + \mathrm{Cov}_\omega(\Theta,\beta)\,\mathcal{E}((1-\beta)\Theta^2)$$

$$+\,\mathrm{Cov}_\omega(\beta,\Theta)\Big(2\mathcal{E}(\Theta)\mathcal{E}(\beta\Theta) - \mathcal{E}(\Theta^2)\mathcal{E}(\beta) - \mathcal{E}(\beta\Theta^2)\Big)$$

In particular, if risk aversion is homogeneous (that is, $\beta = (b, \cdots, b)$) then we get

$$\frac{dr_t}{dW_t} = -b\mathrm{Cov}_\omega(\Theta, R) + b(b-1)\frac{1}{2}\mathrm{Cov}_\omega(\Theta^2,\Theta)$$
$$= -\mathrm{Cov}_\omega(-b\sigma(\delta_i) + 0.5b(b-1)(\delta_i^2) - b\rho_i, (\delta_i)) \quad (25)$$

and the claim follows.

C Proofs for Section 5

Proof of Proposition 5.1

1. We can rewrite the defining identity for the stock price as

$$\int_0^t M_\tau D_\tau d\tau + S_t\, M_t = E_t\left[\int_0^\infty M_\tau D_\tau d\tau\right].$$

Thus,

$$M_t D_t dt + d(S_t M_t) = \gamma_t dB_t$$

for an adapted process γ_t given by

$$\gamma_t = E_t\left[\int_t^\infty \mathcal{D}_t\,(M_\tau D_\tau)\,d\tau\right]$$

where $\mathcal{D}$ denotes the Malliavin derivative[22]. Using Ito's formula, we get

$$\gamma_t = S_t M_t(-\theta_t + \sigma_t^S).$$

[22] For a general presentation of Malliavin derivatives and Malliavin calculus we refer to Nualart (1995). We also refer to Detemple et al. (2005) for specific applications in financial models.

Let us now calculate γ_t. We have

$$\mathcal{D}_t(M_\tau D_\tau) = \mathcal{D}_t(M_\tau)D_\tau + M_\tau \sigma D_\tau$$

and

$$\mathcal{D}_t(M_\tau) = \mathcal{D}_t(F(c_{10}^{\gamma_1}M_{1\tau}, \cdots, c_{N0}^{\gamma_N}M_{N\tau})) = \sum_i c_{i0}^{\gamma_1} F_{a_i} \mathcal{D}_t(M_{i\tau}) = -M_\tau \theta_\tau .$$

Thus,

$$\gamma_t = E_t\left[\int_t^\infty (\sigma - \theta_\tau) M_\tau D_\tau d\tau\right]$$

which gives σ_t^S. The derivation of the upper and lower bounds is straightforward.

2. Since

$$\theta_t = \sum_i \omega_{i\,t}\theta_i,$$

we get

$$\frac{E_t[\theta_{t+u}M_{t+u}D_{t+u}]}{E_t[M_{t+u}D_{t+u}]} - \theta_{I_K} = \sum_{i \neq I_K} \theta_i \, \frac{E_t[\omega_{i\,t+u}M_{t+u}D_{t+u}]}{E_t[M_{t+u}D_{t+u}]}.$$

By (50), we get

$$\omega_{i,t+u} \le \frac{b_i}{\min_i b_i} c_{i\,t+u} D_{t+u}^{-1} \le e^{-\psi_i(t+u)+\eta_i W_{t+u}} \tag{26}$$

for some $\psi_i > 0$. Now, the same argument as in the proof of Proposition 7.2 (based on Lemma A.1) implies that

$$\lim_{t\to\infty} \frac{E_t[e^{-\psi_i(t+u)+\eta_i W_{t+u}} M_{t+u}\, D_{t+u}]}{E_t[e^{-\psi_i(t+u)+\eta_i W_{t+u}} M_{t+u}^{I_K}\, D_{t+u}]} = 1$$

and from the law of large numbers we have

$$\lim_{t\to\infty} \frac{E_t[e^{-\psi_i(t+u)+\eta_i W_{t+u}}\, M_{t+u}\, D_{t+u}]}{E_t[M_{t+u}\, D_{t+u}]} \;=\; 0$$

for all $i \neq I_K$. With (26), this leads to

$$\frac{E_t[\theta_{t+u}M_{t+u}D_{t+u}]}{E_t[M_{t+u}D_{t+u}]} \to \theta_{I_K}. \tag{27}$$

and then to

$$\lim_{t\to\infty} \sigma_t^S = \sigma.$$

3. We have

$$\frac{S_t}{D_t} = E_t\left[\int_t^\infty \frac{M_\tau D_\tau}{M_t D_t} d\tau\right].$$

Note we can rewrite the aggregate consumption condition as

$$1 = \sum_{i=1}^N \left(\frac{M_\tau}{M_t}\right)^{-b_i} \left(\frac{M_{i\tau}}{M_{it}}\right)^{b_i} (c_{it} D_t^{-1}) \tag{28}$$

Then, the same argument as in the proof of Lemma A.1 gives us

$$\left(\sum_i \left((c_{it}D_t^{-1})^{\gamma_i} \frac{M_{i\tau}}{M_{it}}\right)^{1/\gamma}\right)^\gamma \le \frac{M_\tau}{M_t} \le \left(\sum_i \left((c_{it}D_t^{-1})^{\gamma_i} \frac{M_{i\tau}}{M_{it}}\right)^{1/\Gamma}\right)^\Gamma \tag{29}$$

for $\Gamma \ge 1$ such that $\Gamma\, b_i > 1$ for all i and for $\gamma \le 1$ such that $\gamma\, b_i \le 1$ for all i. Similarly, we have

$$\left(\sum_i \left((c_{it}D_t^{-1})^{\gamma_i} E_t\left[\frac{D_\tau M_{i\tau}}{D_t M_{it}}\right]\right)^{1/\gamma}\right)^\gamma \le E_t\left[\frac{D_\tau M_\tau}{D_t M_t}\right]$$
$$\le \left(\sum_i \left((c_{it}D_t^{-1})^{\gamma_i} E_t\left[\frac{D_\tau M_{i\tau}}{D_t M_{it}}\right]\right)^{1/\Gamma}\right)^\Gamma \tag{30}$$

Since all finite dimensional norms are equivalent, there exist constants $K_1 > K_2 > 0$ such that

$$K_2 \sum_i (c_{it}D_t^{-1})^{\gamma_i} E_t\left[\frac{D_\tau M_{i\tau}}{D_t M_{it}}\right] \le E_t\left[\frac{D_\tau M_\tau}{D_t M_t}\right] \le K_1 \sum_i (c_{it}D_t^{-1})^{\gamma_i} E_t\left[\frac{D_\tau M_{i\tau}}{D_t M_{it}}\right]. \tag{31}$$

Pick now an $\epsilon > 0$ and let $T > 0$ be so large that

$$\sum_i \left(\frac{S}{D}\right)_i e^{-T\left(\frac{S}{D}\right)_i^{-1}} < \epsilon$$

then by (31) we have

$$E_t\left[\int_{t+T}^\infty \frac{M_\tau D_\tau}{M_t D_t} d\tau\right] \le K_1 \epsilon$$

for all $t > 0$. Now, the same argument as in the proof of Proposition 7.2 implies that, for any $\theta > 0$,

$$E_t\left[\frac{D_{t+\theta}M_{t+\theta}}{D_tM_t}\right] \to e^{-\theta\left(\frac{S}{D}\right)^{-1}_{I_K}}.$$

Consequently, because of uniform boundedness, guaranteed by (31), we can interchange limit and integration and we have

$$E_t\left[\int_0^T \frac{D_{t+\theta}M_{t+\theta}}{D_tM_t}d\theta\right] \to \int_0^T e^{-\theta\left(\frac{S}{D}\right)^{-1}_{I_K}}\, d\theta.$$

Since $\epsilon > 0$ is arbitrary and T can be made arbitrarily large, we are done.

4. After some algebraic manipulations, we have

$$\frac{S_t}{D_t} = E_t\left[\int_t^{+\infty} e^{-\int_0^t \zeta_s ds}\frac{N_{t+\tau}}{N_t}d\tau\right]$$

where

$$\zeta_t = r_t + \theta_t\sigma - \mu$$

is the risk-adjusted discount rate and

$$N_t = e^{-0.5\int_0^t(\theta_s-\sigma)^2ds-\int_0^t(\theta_s-\sigma)dW_s}$$

is an exponential martingale. Thus, N_t is a density process of a measure $d\nu$ and, under this measure, we can rewrite

$$\frac{S_t}{D_t} = E_t^\nu\left[\int_t^{+\infty} e^{-\int_t^\tau \zeta_s ds}d\tau\right].$$

Lemma 1 in Mele (2007) implies that $\frac{S_t}{D_t}$ is increasing (decreasing) if and only if ζ_t is decreasing (increasing).

Furthermore, if g is such that

$$\frac{S_t}{D_t} = g(t, W_t)$$

then, by the Ito formula,

$$\sigma_t^S = \sigma + \frac{\partial g/\partial W_t}{g}$$

and hence excess volatility is positive if and only if g is monotone increasing. Thus, we need to check monotonicity of the risk-adjusted discount rate ζ_t.

We have

$$\frac{\partial \zeta_t}{\partial W_t} = \frac{\partial r_t}{\partial W_t} + \frac{\partial \theta_t}{\partial W_t}\sigma$$

and

$$\frac{\partial \theta_t}{\partial W_t} = -2b \operatorname{Var}^{\omega_i}(\delta_i)$$

and hence we need that

$$\begin{aligned} \operatorname{Cov}^{\omega_i}(b\sigma\delta_i + 0.5b(1-b)\delta_i^2 + b\rho_i, \delta_i) - 2b\sigma\ \operatorname{Var}^{\omega_i}(\delta_i) \\ = \operatorname{Cov}^{\omega_i}(0.5b(1-b)\delta_i^2 + b\rho_i - b\sigma\delta_i, \delta_i). \end{aligned} \quad (32)$$

Proof of Proposition 5.2

Introduce a new measure $dP^{(\alpha)}$ such that its restriction on $\mathcal{F}_T$ is given by

$$dP_T^{(\alpha)} \;=\; \frac{D_T^\alpha}{E[D_T^\alpha]}\, dP_T$$

where P_T is the physical measure. Under the measure $dP^{(\alpha)}$, W_t has drift $\sigma\alpha$, $dW_t = dW_t^{(\alpha)} + \sigma^2\alpha dt$, and D_t has a drift $\mu + \sigma^2\alpha$. By the Kolmogorov extension theorem, we can extend this measure to the set of infinite paths. Then, we can rewrite the expression for $R(t,T)$ as

$$R(t,T) = \frac{E_t[D_T]}{D_t}\frac{D_t}{S_t}E_t^{(1)}\left[\frac{S_T}{D_T}\, e^{\int_t^T D_s S_s^{-1}\, ds}\right] \quad (33)$$

By (31) we have,

$$L_2 \equiv K_2\, n^{-\max_i \gamma_i}\ \min_i\, S_i\ \le \frac{S_t}{D_t} \le K_1 \sum_i S_i \equiv L_1.$$

Consequently, S_t/D_t is uniformly bounded both from zero and infinity and hence

$$\begin{aligned} \mu\ +\ \Big(\log(L_2/L_1)\ +\ \log E_t^{(1)}\left[e^{\int_t^T D_s S_s^{-1}\, ds}\right]\Big) \\ \le\ \liminf_{T\to\infty}\, (T-t)^{-1}\, \log\, R(t,T) \\ \le\ \mu\ +\ (T-t)^{-1}\Big(\log(L_1/L_2)\ +\ \log E_t^{(1)}\left[e^{\int_t^T D_s S_s^{-1}\, ds}\right]\Big) \end{aligned} \quad (34)$$

and thus

$$
\begin{aligned}
\mu \;+\; \liminf_{T\to\infty}\,(T-t)^{-1}\log E_t^{(1)}\left[e^{\int_t^T\, D_s S_s^{-1}\,ds}\right] \\
\le\; \liminf_{T\to\infty}\,(T-t)^{-1}\,\log\, R(t,T) \;\le\; \limsup_{T\to\infty}\,(T-t)^{-1}\,\log\, R(t,T) \\
\le\; \mu \;+\; \limsup_{T\to\infty}\,(T-t)^{-1}\log E_t^{(1)}\left[e^{\int_t^T\, D_s S_s^{-1}\,ds}\right]. \quad (35)
\end{aligned}
$$

The optimal consumption of agent i can be rewritten as

$$
c_{it} = e^{-\rho_i^{(\alpha)} b_i t} M_t^{-b_i} (Z_{it}^{(\alpha)})^{b_i} c_{i0}
$$

where

$$
\rho_i^{(\alpha)} = \rho_i - \delta_i \sigma^2 \alpha \qquad \text{and} \qquad Z_{it}^{(\alpha)} \;=\; e^{\delta_i W_t^{(\alpha)} - \frac{1}{2}\delta_i^2 t}.
$$

Thus, under this new measure everything looks the same, apart from the fact that agents have different discount rates. Note that the stock price is still calculated under the original, physical measure, but we can rewrite it as

$$
\begin{aligned}
\frac{S_t}{D_t} \;&=\; \frac{1}{M_t D_t} E_t\left[\int_t^\infty M_\tau D_\tau d\tau\right] = \frac{1}{M_t D_t}\int_t^\infty E_t[D_\tau^\alpha]\frac{E_t\left[\,M_\tau\, D_\tau^{1-\alpha} D_t^\alpha\right]}{E_t[D_\tau^\alpha]} d\tau \\
&= \frac{1}{M_t D_t}\int_t^\infty E_t[D_\tau^\alpha]\, E_t^{(\alpha)}\left[M_\tau\, D_\tau^{1-\alpha}\right] d\tau \quad (36)
\end{aligned}
$$

We define agent $A(\alpha)$ as (being the analog of agent I_K)

$$
(\rho_A^{(\alpha)} + \gamma_A(\mu + \sigma^2\alpha - 0.5\sigma^2) + 0.5\delta_A^2) \;=\; \min_i\,(\rho_i^{(\alpha)} + \gamma_i(\mu + \sigma^2\alpha - 0.5\sigma^2) + 0.5\delta_i^2) \quad (37)
$$

By Corollary 6.1, under the measure $P^{(\alpha)}$, we have

$$
M_t \sim c_{I_K\,0}^{\gamma_{A(\alpha)}} e^{-\rho_{A(\alpha)}\, t} Z_{A(\alpha)\,t} D_t^{-\gamma_{A(\alpha)}}.
$$

Similarly, in complete analogy with the proof of Proposition 5.1,

$$
E_t^{(\alpha)}\left[M_{t+u}\, D_{t+u}^{1-\alpha}\right] \;\sim\; E_t^{(\alpha)}\left[M_{t+u}^{(\alpha)}\, D_{t+u}^{1-\alpha}\right]
$$

under $P^{(\alpha)}$ and we obtain that

$$
\lim_{t\to\infty}\frac{S_t}{D_t} = S_{A(\alpha)} \qquad P^{(\alpha)} - a.s.
$$

by the same argument as in the proof of Proposition 5.1.

Now, Jensen's inequality implies

$$\log E_t\left[S_T\, e^{\int_t^T D_s S_s^{-1}\,ds}\right] \;=\; \log E_t[D_T^\alpha] + \log \frac{E_t\left[D_T^\alpha\, D_T^{-\alpha}\, S_T\, e^{\int_t^T D_s S_s^{-1}\,ds}\right]}{E_t[D_T^\alpha]}$$

$$=\; \log E_t[D_T^\alpha] \;+\; \log\, E_t^\alpha\left[\left(D_T^{1-\alpha}\,\frac{S_T}{D_T}\, e^{\int_t^T D_s S_s^{-1}\,ds}\right)\right]$$

$$\geq\; E_t^\alpha[\log(S_T/D_T)] + \int_t^T E_t^{(\alpha)}[(D_s/S_s)]\,ds + E_t^{(\alpha)}[\log(D_T^{1-\alpha})] + \log E_t[D_T^\alpha] \tag{38}$$

We have

$$\log E_t[D_T^\alpha] \;=\; E_t[e^{\alpha((\mu-0.5\sigma^2)T+\sigma W_T)}]$$
$$=\; \log\left(e^{\sigma\,\alpha\,W_t}\, e^{0.5(1-\lambda)T\,\alpha^2\sigma^2 + T\,\alpha(\mu-0.5\sigma^2)}\right)$$
$$=\; \sigma\,\alpha\,W_t \;+\; T\,\alpha\left(\mu+0.5\sigma^2((1-\lambda)\alpha-1)\right) \tag{39}$$

and

$$E_t^{(\alpha)}[\log(D_T^{1-\alpha})] \;=\; E_t^\alpha[\log(e^{(1-\alpha)\,((\mu+(\alpha-0.5)\sigma^2)T+\sigma W_T^{(\alpha)}})]$$
$$=\; (1-\alpha)\,((\mu+(\alpha-0.5)\sigma^2)T \;+\; (1-\alpha)\sigma\,(W_t-\sigma^2\alpha\,\lambda\,T) \tag{40}$$

Since S_T/D_T converges to $S_{A(\alpha)}$ $P^{(\alpha)}$−almost surely, a slight modification of the proof of Proposition 5.1 implies that in fact

$$\lim_{T\to\infty}(1-\lambda)^{-1}\,T^{-1}\int_{\lambda T}^T E_{\lambda T}^{(\alpha)}[(D_s/S_s)]ds \;=\; S_{A(\alpha)}^{-1}$$

and

$$\lim_{\to\infty} T^{-1}\,E_t^\alpha[\log(S_T/D_T)] \;=\; 0.$$

and, by the law of large numbers,

$$\lim_{T\to\infty} T^{-1}\log S_t \;=\; \lim_{T\to\infty}\log D_t \;=\; \lambda\,(\mu-0.5\sigma^2).$$

Hence,

$$\liminf_{T\to\infty} T^{-1}\log\, R(t,T) \;\geq\; -\,\lambda\,(\mu-0.5\sigma^2)$$
$$+\;\alpha\left(\mu+0.5\sigma^2((1-\lambda)\alpha-1)\right) \;+\; (1-\alpha)\,((\mu+(\alpha-0.5)\sigma^2) \;+\; (1-\lambda)\,S_{A(\alpha)}^{-1} \tag{41}$$

Proof of Proposition 5.3

1. By the budget constraint, the wealth is given by the present value of future consumption, that is

$$M_t w_{it} \ = E_t \left[\int_t^\infty M_\tau c_{i\tau} d\tau \right] \ = c_{i0} E_t \left[\int_t^\infty e^{-\rho_i b_i \tau} M_\tau^{1-b_i} e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} d\tau \right].$$

Similarly to what we did with the stock price (Proof of Proposition (5.1), we get that

$$\int_0^t M_\tau c_{i\tau} d\tau + M_t w_{it} = E_t \left[\int_0^\infty M_\tau c_{i\tau} d\tau \right]$$

is a martingale and hence

$$M_t c_{it} dt + w_{it} dM_t + M_t dw_{it} + d\langle M_t, w_{it} \rangle = \gamma_t dW_t$$

where, by the Clark-Ocone formula,

$$\gamma_t = E_t \left[\mathcal{D}_t (M_\tau c_{i\tau}) d\tau \right] .$$

By the Ito's formula,

$$\gamma_t = w_{it} M_t (-\theta_t + \sigma_t \pi_{it})$$

Now, by the rules for Malliavin derivatives,

$$\begin{aligned}
&\mathcal{D}_t (M_\tau c_{i\tau}) = \mathcal{D}_t \left(e^{-\rho_i b_i \tau} M_\tau^{1-b_i} e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} \right) \\
&= (1-b_i) e^{-\rho_i b_i \tau} M_\tau^{-b_i} \mathcal{D}_t (M_\tau) e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} + e^{-\rho_i b_i \tau} M_\tau^{1-b_i} b_i \delta_i e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} \\
&= -(1-b_i) e^{-\rho_i b \tau} M_\tau^{-b_i} \theta_\tau M_\tau e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} + e^{-\rho_i b_i \tau} M_\tau^{1-b_i} b_i \delta_i e^{b_i(\delta_i W_\tau - \frac{1}{2}\delta_i^2 \tau)} \\
&\qquad = (b_i \delta_i + (b_i - 1)\theta_\tau) M_\tau c_{i\tau}
\end{aligned} \tag{42}$$

Thus,

$$\gamma_t = E_t \left[\int_t^\infty (b_i \delta_i + (b_i - 1)\theta_\tau) M_\tau c_{i\tau} d\tau \right]$$

and we get

$$\sigma_t \pi_{it} = \theta_t + \frac{E_t \left[\int_t^\infty (b_i \delta_i + (b_i - 1)\theta_\tau) M_\tau c_{i\tau} d\tau \right]}{E_t \left[\int_t^\infty M_\tau c_{i\tau} d\tau \right]}.$$

2. Suppose now that $\gamma_i > 1$. Let $\Gamma \geq 1$ be such that $\Gamma\, b_j/(1-b_i) > 1$ for all j and $\gamma \leq 1$ be such that $\gamma\, b_j/(1-b_i) \leq 1$ for all j. Then, the same argument as in the proof of Lemma A.1 gives us

$$\left(\sum_j \left(c_{j0}^{\gamma_j(1-b_i)} E_t[e^{-\rho_j(1-b_i)T} Z_{jT}^{1-b_i} Z_{iT}^{b_i} D_T^{-\gamma_j(1-b_i)}]\right)^{1/\gamma}\right)^{\gamma} \leq E_t[M_T^{1-b_i} Z_{iT}^{b_i}]$$
$$\leq \left(\sum_j \left(c_{j0}^{\gamma_j(1-b_i)} E_t[e^{-\rho_j(1-b_i)T} Z_{jT}^{1-b_i} Z_{iT}^{b_i} D_T^{-\gamma_j(1-b_i)}]\right)^{1/\Gamma}\right)^{\Gamma}. \quad (43)$$

Denote

$$c_{it}^{(I_K)} \;=\; e^{-\rho_i\, b_i\, \tau}\, (M_\tau^{(I_K)})^{-b_i}\, Z_{it}^{b_i}.$$

Note that

$$\frac{E_t[M_{t+u}^{(I_K)}\, c_{i\,t+u}^{I_K}]}{M_t^{(I_K)}\, c_{i\,t}^{(I_K)}}$$
$$=\; e^{-(\rho_i\, b_i + \rho_{I_K}(1-b_i))u}\, E_t\left[\left(\left(\frac{D_{t+u}}{D_t}\right)^{-\gamma_{I_K}} \frac{Z_{I_K\,t+u}}{Z_{I_K\,t}}\right)^{1-b_i} \left(\frac{Z_{i\,t+u}}{Z_{i\,t}}\right)^{b_i}\right] \quad (44)$$

is independent of t.

By Lemma A.1, we have

$$M_t\, c_{i\,t} \;\sim\; M_t^{(I_K)}\, c_{i\,t}^{(I_K)}.$$

and a direct application of (43) and the same argument as in the proof of Proposition 7.2 implies that

$$E_t[M_{t+u}\, c_{i\,t+u}] \;\sim\; E_t[M_{t+u}^{(I_K)}\, c_{i\,t+u}^{(I_K)}].$$

By the same argument as in the proof of (27) (but based on the bounds of 43), we get that

$$\lim_{t\to\infty} \frac{E_t[\omega_{j\,t+u}\, D_{t+u}^{-1}\, M_{t+u}\, c_{i\,t+u}]}{E_t[M_{t+u}\, c_{i\,t+u}]} \;\to\; 0$$

and in complete analogy with the proof of (27), we obtain

$$E_t[\theta_{t+u} M_{t+u} c_{i\,t+u}] \sim \theta^{(I_K)} E_t[M_{t+u} c_{i\,t+u}].$$

and from there

$$\lim_{t\to\infty} \frac{E_t[\theta_{t+u} M_{t+u} c_{i\,t+u}]}{M_t\, c_{i\,t}} = \theta^{(I_K)} \frac{E_t[M_{t+u}^{(I_K)}\, c_{i\,t+u}^{(I_K)}]}{M_t^{(I_K)}\, c_{i\,t}^{(I_K)}}$$

and

$$\lim_{t\to\infty} \frac{E_t[M_{t+u}\, c_{i\,t+u}]}{M_t\, c_{i\,t}} = \frac{E_t[M_{t+u}^{(I_K)}\, c_{i\,t+u}^{(I_K)}]}{M_t^{(I_K)}\, c_{i\,t}^{(I_K)}}.$$

Now, let us prove that

$$\lim_{t\to\infty} \int_0^\infty \frac{E_t[\theta_{t+u}\, M_{t+u}\, c_{i\,t+u}]}{M_t\, c_{i\,t}} du = \theta^{(I_K)} \int_0^\infty \frac{E_t[M_{t+u}\, c_{i\,t+u}^{(I_K)}]}{M_t\, c_{i\,t}^{(I_K)}} du$$

and

$$\lim_{t\to\infty} \frac{W_{it}}{c_{it}} = \lim_{t\to\infty} \int_0^\infty \frac{E_t[M_{t+u}\, c_{i\,t+u}]}{M_t\, c_{i\,t}} du = \int_0^\infty \frac{E_t[M_{t+u}\, c_{i\,t+u}^{(I_K)}]}{M_t\, c_{i\,t}^{(I_K)}} du$$

By the Lebesgue dominated convergence theorem, it suffices to show that there exists an integrable function $g(u)$ such that

$$\frac{E_t[M_{t+u}\, c_{i\,t+u}]}{M_t\, c_{i\,t}} = E_t\left[\left(\frac{M_{t+u}}{M_t}\right)^{1-b_i} \left(e^{-\rho\, u} \frac{Z_{i\,t+u}}{Z_{it}}\right)^{b_i}\right] \le g(u).$$

By (29) and using the fact that for $\alpha > 0$, there exists a constant $K > 0$ such that

$$\left(\sum_i x_i\right)^\alpha \le K \sum_i x_i^\alpha,$$

we have

$$\left(\frac{M_\tau}{M_t}\right)^{1-b_i} \left(e^{-\rho_i(\tau-t)} \frac{Z_{i\,\tau}}{Z_{it}}\right)^{b_i}$$
$$\le K \sum_j e^{-\rho_j(\tau-t)(1-b_i)} \left(\frac{Z_{j\tau}}{Z_{jt}} \left(\frac{D_\tau}{D_t}\right)^{-\gamma_j}\right)^{1-b_i} \left(e^{-\rho_i(\tau-t)} \frac{Z_{i\,\tau}}{Z_{it}}\right)^{b_i}. \quad (45)$$

Now, using the Young inequality

$$x^{1-b_i} y^{b_i} \le (1-b_i)x + b_i y$$

we get that

$$e^{-\rho_j(\tau-t)(1-b_i)}\left(\frac{Z_{j\tau}}{Z_{jt}}\left(\frac{D_\tau}{D_t}\right)^{-\gamma_j}\right)^{1-b_i}\left(e^{-\rho_i(\tau-t)}\frac{Z_{i\tau}}{Z_{it}}\right)^{b_i}$$
$$=\left(e^{-\rho_j(\tau-t)}\left(\frac{Z_{j\tau}}{Z_{jt}}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_j}\right)^{1-b_i}\left(e^{-\rho_i(\tau-t)}\frac{Z_{i\tau}}{Z_{it}}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_i}\right)\right)^{b_i}$$
$$\leq K\left(e^{-\rho_i(\tau-t)}\frac{Z_{i\tau}}{Z_{it}}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_i}+e^{-\rho_j(\tau-t)}\frac{Z_{j\tau}}{Z_{jt}}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_j}\right) \quad (46)$$

and hence

$$\frac{E_t[M_{t+u}\,c_{i\,t+u}]}{M_t\,c_{i\,t}} \leq K\sum_j e^{-u\,S_i^{-1}}$$

which is integrable by the assumption that $S_i > 0$. As a direct consequence, we have

$$\lim_{t\to\infty}\pi_{it} = \frac{\delta_i + \theta^{(I_K)}}{\sigma\,\gamma_i}.$$

3. We define

$$f(t, W_t) = E_t\left[\int_t^\infty e^{-\rho_i b_i(\tau-t)}\frac{Z_{i\tau}^{b_i}\,M_\tau^{1-b_i}}{Z_{it}^{b_i}\,M_t^{1-b_i}}\right].$$

Then, the wealth w_{it} of agent i satisfies

$$w_{it} = Z_{it}^{b_i}M_t^{-b_i}e^{-\rho_t}f(t, W_t).$$

An applucation of Ito's formula implies that

$$\pi_{it}\sigma_t = \delta_i b_i + \theta_t b_i + \frac{(d/dW_t)f(t, W_t)}{f(t, W_t)}$$

and hence

$$\pi_{it}^{\text{hedging}} = \sigma_t^{-1}\,\frac{(d/dW_t)f(t, W_t)}{f(t, W_t)}.$$

To determine the sign of the hedging portfolio, we need to check whether f is monotone increasing.

We have

$$E_t\left[\int_t^\infty e^{-\rho_i b_i(\tau-t)}\,\frac{Z_{i\tau}^{b_i}\,M_\tau^{1-b_i}}{Z_{it}^{b_i}\,M_t^{1-b_i}}\right]$$

$$= E_t\left[e^{-\rho_i b_i(\tau-t)\,-\,0.5\,b_i\,\delta_i^2(\tau-t)+\delta_i b_i(W_\tau-W_t)\,-(1-b_i)\int_t^\tau (r_s+0.5\theta_s^2)ds+\int_t^\tau(\delta_i b_i+(b_i-1)\theta_s)dW_s}\right]$$

$$= E_t^{P_i}\left[\int_t^\infty e^{-\int_t^\tau \zeta_{is}ds}\,d\tau\right] \quad (47)$$

where P_i is a new measure with density process

$$\frac{dP_i}{dP} = e^{-\int_0^t(\delta_i b_i+(b_i-1)\theta_s)^2 ds+\int_t^\tau(\delta_i b_i+(b_i-1)\theta_s)dW_s}$$

and

$$\begin{aligned}\zeta_{it} &= \rho_i b_i+0.5b_i\delta_i^2+(1-b_i)r_t+0.5(1-b_i)\theta_t^2-0.5(\delta_i b_i+(b_i-1)\theta_t)^2\\ &= \rho_i b_i+0.5b_i(1-b_i)\delta_i^2+(1-b_i)r_t+0.5(1-b_i)b_i\,\theta_t^2+\delta_i b_i(1-b_i)\theta_t \quad (48)\end{aligned}$$

Thus, it all reduces to checking monotonicity of

$$\zeta_i(W_t)=r_t+\frac{1}{2}b_i\theta_t^2+\delta_i b_i\theta_t$$

Suppose that risk aversion is homogeneous. Then, the derivative of ζ_i is given by

$$\mathrm{Cov}^{\omega_i}(b\sigma\delta_i+\frac{1}{2}b(1-b)\delta_i^2+b\rho_i,\delta_i)-2b^2(\theta_t+\delta_i)\,\mathrm{Var}^{\omega_i}(\delta_i)$$

Thus, we get the result (using the fact that $\theta_t\in[\min_j\theta_j,\max_j\theta_j]$).

D Proofs for Section 6

Proof of Proposition 6.1

We easily get from Equation (17) that

$$\min_{1\le i\le N} M_i \le M \le \max_{1\le i\le N} M_i. \quad (49)$$

The second point is immediate. Note that $F(a_1,\cdots,a_N)>F(a_1',\cdots,a_N')$ whenever $a_i>a_i'$ for $i=1,\cdots N$. The state price density M_t is then decreasing in W_t whenever all the state price densities M_{it} are decreasing in W_t for $i=1,\cdots N$. The long run results are immediate.

Proof of Corollary 6.1

By Lemma A.1, $M_t \geq c_{I_K 0}^{\gamma_{I_K}} e^{-\rho_{I_K} t} Z_{I_K t} D_t^{-\gamma_{I_K}}$. Let, for each k, $\tilde{\gamma}_k = \gamma_k - \delta_k \sigma^{-1}$. Then, for $i \neq I_K$

$$
\begin{aligned}
c_{it} D_t^{-1} &= e^{-\rho_i b_i t} M_t^{-b_i} Z_{it}^{b_i} c_{i0} \leq e^{-\rho_i b_i t} \left(c_{I_K 0}^{\gamma_{I_K}} e^{-\rho_{I_K} t} Z_{I_K t} D_t^{-\gamma_{I_K}} \right)^{-b_i} Z_{it}^{b_i} c_{i0} D_t^{-1} \\
&= \left(\frac{e^{-\rho_i t} c_{i0}^{\gamma_i} Z_{it}}{e^{-\rho_{I_K} t} c_{I_K 0}^{\gamma_{I_K}} Z_{I_K t}} D_t^{\gamma_{I_K} - \gamma_i} \right)^{b_i} = e^{b_i (\kappa_{I_K} - \kappa_i) t + b_i (\tilde{\gamma}_{I_K} - \tilde{\gamma}_i) \sigma W_t}. \quad (50)
\end{aligned}
$$

By definition, $\kappa_{I_K} - \kappa_i < 0$ and therefore $c_{it} D_t^{-1}$ converges to zero by the law of large numbers. Since $\sum_i c_i = D$, we have $\lim_{t \to \infty} c_{I_K t} D_t^{-1} = 1$. The limits when W_t goes to $+\infty$ or $-\infty$ result directly from Corollary 6.1.

Recall that the relative level of risk tolerance is given by $\omega_{it} \equiv \frac{b_i c_{it}}{\sum_{j=1}^N b_j c_{jt}}$. Differentiating the equation $\sum_{i=1}^N F^{-b_i} a_i^{b_i} = 1$, we get $F_{a_i}(c_{10}^{\gamma_1} M_1, ..., c_{N0}^{\gamma_N} M_N) = \frac{M_t}{c_{i0}^{\gamma_i} M_i} \omega_{it}$ hence the relative level of risk tolerance can be written in the form

$$
\omega_{it} = c_{i0}^{\gamma_i} M_i F^{-1} F_{a_i}
$$

and then

$$
\begin{aligned}
\frac{d\omega_{it}(W_t)}{dW_t} = c_{i0}^{\gamma_i} \frac{dM_{it}}{dW_t} F^{-1} F_{a_i} - c_{i0}^{\gamma_i} M_i F^{-2} F_{a_i} \sum_j F_{a_j} c_{j0}^{\gamma_j} \frac{dM_{jt}}{dW_t} \\
+ c_{i0}^{\gamma_i} M_i F^{-1} \sum_j F_{a_i a_j} c_{j0}^{\gamma_j} \frac{dM_{jt}}{dW_t}. \quad (51)
\end{aligned}
$$

If we differentiate the formula

$$
\sum_k F^{1-b_k} a_k^{b_k} - F = 0
$$

with respect to a_i and then with respect to a_j, we get

$$
\begin{aligned}
F_{a_i a_j} = ((1 - b_j) + (1 - b_i)) \, \omega_{it} \omega_{jt} M_t (c_{i0}^{\gamma_i} M_{it} c_{j0}^{\gamma_j} M_{jt})^{-1} \\
- M_t^{-1} \sum_k (1 - b_k) \omega_{kt} F_{a_i} F_{a_j} + \delta_{ij} (b_i - 1)(c_{i0}^{\gamma_i} M_{it})^{-2} M_t \omega_{it} \quad (52)
\end{aligned}
$$

where δ_{ij} is equal to 1 for $i = j$ and to 0 for $i \neq j$. Replacing in (51) and recombining the different terms leads to

$$
\frac{d\omega_{it}(W_t)}{dW_t} = \omega_{it} \left[b_i (\theta_t - \theta_i) - \sum_j \omega_{jt} b_j (\theta_t - \theta_j) \right]. \quad (53)
$$

E Proofs for Section 7

Proof of Proposition 7.1

We have $E_t\left[M_{iT}\right] = \exp(-r_i(T-t)-\kappa_i t-\theta_i W_t)$. We obtain as a corollary to Equation (15) that

$$\left(\sum_{i=1}^{N} c_{i0}^{\gamma_i/\gamma} E_t\left[M_{iT}\right]^{1/\gamma}\right)^{\gamma} \leq E_t\left[M_T\right] \leq \left(\sum_{i=1}^{N} c_{i0}^{\gamma_i/\Gamma} E_t\left[M_{iT}\right]^{1/\Gamma}\right)^{\Gamma} \tag{54}$$

From this we obtain $\lim_{T\to\infty} \frac{1}{T} \ln E_t\left[M_T\right] = r_{I_0}$ and since $B(t,T) = \frac{1}{M_t} E_t\left[M_T\right]$, we get

$$\lim_{T\to\infty} Y(t,T) = r_{I_0}.$$

Proof of Proposition 7.2

For $\lambda \in (\lambda_{j-1}, \lambda_j)$ and $t = \lambda T$ we have

$$\lim_{T\to\infty} \frac{E_t[M_{iT}]}{E_t[M_{I_j T}]} \to 0$$

for $i \neq I_j$. The long run behavior of $E_t[M_{\alpha t}]$ for $\alpha \in \left(\frac{1}{\lambda_{j+1}}, \frac{1}{\lambda_j}\right)$ derives directly from there and from Equation (54). By definition, $B(t,T)$ is equal to $\frac{1}{M_t} E_t\left[M_T\right]$ which gives the long run behavior of $B(t, \alpha t)$.

Since $\frac{B_t}{t}$ converges to 0 almost surely we have, for $\alpha \in (\frac{1}{\lambda_j}, \frac{1}{\lambda_{j-1}})$,

$$\lim_{t\to\infty} Y(t,\alpha t) = \frac{1}{\alpha - 1}[\kappa_{I_K} - \alpha l_{I_j}\left(1/\alpha\right)]$$

and the long run instantaneous forward rate at date αt seen from date t is given by r_{I_j}.

As far as the uniform convergence is concerned, note that

$$\frac{d}{d\alpha} Y(t,\alpha t) = \frac{t}{(\alpha t - t)^2} \ln B(t,\alpha t) - \frac{1}{\alpha t - t} \frac{\frac{d}{d\alpha} B(t,\alpha t)}{B(t,\alpha t)}.$$

Let $\tilde{P}$ be the equivalent martingale measure, corresponding to M_t. We have $B(t,\alpha t) = E_t^{\tilde{P}}\left[\exp - \int_t^{\alpha t} r_s ds\right]$ and $\frac{d}{d\alpha} B(t,\alpha t) = E_t^{\tilde{P}}\left[-t r_{\alpha t} \exp - \int_t^{\alpha t} r_s ds\right]$. If we denote by Q_t the probability defined by its density $\frac{dQ_t}{d\tilde{P}} = \frac{\exp - \int_t^{\alpha t} r_s ds}{E\left[\exp - \int_t^{\alpha t} r_s ds\right]}$ then we have $\frac{d}{d\alpha} Y(t,\alpha t) = \frac{1}{(\alpha - 1)}\left(-Y(t,\alpha) + E_t^{Q_t}\left[r_{\alpha t}\right]\right)$.

We also have

$$
\left(\sum_i c_{i0}^{\gamma_i/\gamma}\right)^{\frac{1}{\gamma}} \exp\left[(\alpha t - t)\min_i l_i(0)\right]
$$

$$
\leq B(t,\alpha t) \leq \left(\sum_i c_{i0}^{\gamma_i/\Gamma}\right)^{\frac{1}{\Gamma}} \exp\left[(\alpha t - t)\max_i l_i(0)\right]
$$

$$
\frac{1}{t-\alpha t}\frac{1}{\gamma}\ln\left(\sum_i c_{i0}^{\gamma_i/\gamma}\right) - \max_i l_i(0)
$$

$$
\leq Y(t,\alpha t) \leq \frac{1}{t-\alpha t}\frac{1}{\Gamma}\ln\left(\sum_i c_{i0}^{\gamma_i/\Gamma}\right) - \min_i l_i(0)
$$

which gives us that $\frac{1}{\alpha-1}Y(t,\alpha t)$ is bounded on the compact subsets of $(1,\infty)$. Using the expression for r_t given by Proposition 4.1 we also have that $r_{\alpha t}$ and hence $\frac{1}{\alpha-1}E_t^{Q_t}[r_{\alpha t}]$ are bounded on the compact subsets of $(1,\infty)$. The mappings $\alpha \to Y(t,\alpha t)$ are then uniformy Lipschitz on the compact subsets of $(1,\infty)$ and the convergence of $Y(t,\alpha t)$ to $Y(\alpha)$ is then uniform on the compact subsets of $(1,\infty)$.

Proof 1 (Proposition 7.3) *Let $\lambda \in (\lambda_j\,,\,\lambda_{j+1})$. Clearly, it suffices to show that*

$$
\frac{E_{\lambda T}[c_{iT}\,D_T^{-1}\,M_T]}{E_{\lambda T}[M_T]} \;\to\; 0 \tag{55}
$$

for all $i \neq I(j)$. Indeed, since

$$
\sum_k c_{kT}\,M_T \;=\; 1,
$$

it immediately follows that agent $I(j)$ is the only one surviving in the long run.

Now, by Proposition 7.2,

$$
E_{\lambda T}[M_T] \;\sim\; c_{I_j 0}^{\gamma_{I_j}} E_{\lambda T}\left[M_{I_j \alpha t}\right] \tag{56}
$$

for $\lambda \in (\lambda_j, \lambda_{j+1})$. Furthermore, by (49),

$$
M_T \;\leq\; \max_k M_{kT} \;\leq\; \sum_k M_{kT}
$$

and therefore

$$E_{\lambda T}[c_{iT}\,D_T^{-1}\,M_T]\;\le\;\sum_k E_{\lambda T}[c_{iT}\,D_T^{-1}\,M_{kT}]\,.$$

Therefore, by (56), the required assertion (55) will follow if we prove that

$$\lim_{T\to\infty}\frac{E_{\lambda T}[c_{iT}\,D_T^{-1}\,M_{kT}]}{E_{\lambda T}\left[M_{I_j\alpha t}\right]}\;=\;0 \tag{57}$$

for all $i \neq I(j)$ and all k. We will consider two different cases.

Case 1. *$k \neq I(j)$. In this case, the trivial bound $c_{iT}\,D_T^{-1}\;\le\;1$ implies*

$$\frac{E_{\lambda T}[c_{iT}\,D_T^{-1}\,M_{kT}]}{E_{\lambda T}\left[M_{I_j\alpha t}\right]}\;\le\;\frac{E_{\lambda T}[M_{kT}]}{E_{\lambda T}\left[M_{I_j\alpha t}\right]}$$

and (57) follows from (11).

Case 2. *$k = I(j)$. In this case, the same argument as in (50) together with the market clearing condition implies that*

$$c_{iT}\,D_T^{-1}\;\le\;\min\left\{e^{b_i(\kappa_{I(j)}-\kappa_i)t+b_i(\theta_{I(j)}-\theta_i)W_T}\,,\,1\right\}\,,$$

Then, for $t = \lambda T$, we get

$$\begin{aligned}\frac{E_t[\,c_{i\,T}\,D_T^{-1}\,M_{I(j)T}]}{E_t[M_{I(j)T}]}\;&=\;\frac{E_t[\,c_{i\,T}\,D_T^{-1}\,e^{(\delta_i-\gamma_i\sigma)W_t}]}{E_t[e^{(\delta_i-\gamma_i\sigma)W_t}]}\\ \le\;E_{\lambda T}\Bigg[&e^{-\theta_{I(j)}(W_T-W_{\lambda T})\,-\,0.5\,(\theta_{I(j)})^2(1-\lambda)T}\\ \times\min&\left\{e^{b_i(\kappa_{I(j)}-\kappa_i)T+b_i(\theta_{I(j)}-\theta_i)W_T}\,,\,1\right\}\Bigg]\,.\end{aligned} \tag{58}$$

Denote $\eta\;=\;b_i(\kappa_{I(j)}-\kappa_i)$, $\zeta\;=\;b_i\,(\theta_{I(j)}-\theta_i)$ and $C_t\;=\;\zeta\,W_{\lambda T}$. We need to consider the cases $\zeta > 0$ and $\zeta < 0$ separately.

If $\zeta > 0$ *that is* $\theta_{I(j)} > \theta_i$ *, then,*

$$
\begin{aligned}
& E_{\lambda T}\left[e^{-\theta_{I(j)}\,(W_T - W_{\lambda T}) \,-\, 0.5\,(\theta_{I(j)})^2 (1-\lambda) T} \times \min\left\{ e^{b_i(\kappa_{I(j)} - \kappa_i) T + b_i(\theta_{I(j)} - \theta_i) W_T} \,,\, 1 \right\} \right] \\
&= \; E_{\lambda T}\left[e^{-\theta_{I(j)}(W_T - W_{\lambda T}) \,-\, 0.5\theta^2_{I(j)}(1-\lambda)T} \;\min\left\{ e^{C_t \,+\, \zeta(W_T - W_{\lambda T}) \,+\, \eta\,T},\; 1 \right\} \right] \\
&= \; \frac{1}{\sqrt{2\pi\,(1-\lambda)\,T}} \\
&\times \int_{-\infty}^{-\zeta^{-1}(\eta\,T + C_t)} e^{-x^2/(2T(1-\lambda))}\; e^{-\theta_{I(j)}x \,-\, 0.5\theta^2_{I(j)}(1-\lambda)T}\; e^{C_t \,+\, \zeta\,x \,+\, \eta T}\, dx \\
&+ \; \frac{1}{\sqrt{2\pi\,(1-\lambda)\,T}} \\
&\times \int_{-\zeta^{-1}(\eta\,T + C_t)}^{+\infty} e^{-x^2/(2T(1-\lambda))}\; e^{-\theta_{I(j)}x \,-\, 0.5\,\theta^2_{I(j)}(1-\lambda)T}\, dx \\
&= \; e^{C_t \,+\, (\eta - 0.5\theta^2_{I(j)}(1-\lambda))T}\; \frac{1}{\sqrt{2\pi}} \\
&\times \int_{-\infty}^{-\zeta^{-1}(\eta\,T + C_t)((1-\lambda)T)^{-1/2}} e^{-x^2/2}\; e^{x\,(-\theta_{I(j)} + \zeta)\,((1-\lambda)T)^{1/2}}\, dx \\
&+ \; e^{-\,0.5\theta^2_{I(j)}(1-\lambda)T}\; \frac{1}{\sqrt{2\pi}} \int_{-\zeta^{-1}(\eta\,T + C_t)((1-\lambda)T)^{-1/2}}^{+\infty} e^{-x^2/2}\; e^{-\theta_{I(j)}x((1-\lambda)T)^{1/2}}\, dx
\end{aligned}
\tag{59}
$$

Now, using the identity

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, e^{\nu x}\, dx \;=\; e^{\nu^2/2}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-(x-\nu)^2/2}\, dx \;=\; e^{\nu^2/2}\, N(y - \nu)
$$

where

$$
N(y) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx
$$

is the cdf of the normal distribution, we get

$$
\begin{aligned}
& e^{C_t + (\eta - 0.5\theta^2_{I(j)}(1-\lambda))T} \frac{1}{\sqrt{2\pi}} \\
& \times \int_{-\infty}^{-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2}} e^{-x^2/2} e^{x\,(-\theta_{I(j)}+\zeta)\,((1-\lambda)T)^{1/2}} dx \\
& + \; e^{-\,0.5\theta^2_{I(j)}(1-\lambda)T} \frac{1}{\sqrt{2\pi}} \int_{-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2}}^{+\infty} e^{-x^2/2} e^{-\theta_{I(j)}x((1-\lambda)T)^{1/2}} dx \\
& = \; e^{C_t + (\eta - 0.5\theta^2_{I(j)}(1-\lambda))T} e^{0.5\,(-\theta_{I(j)}+\zeta)^2\,(1-\lambda)T} \\
& \times N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} \; - \; (-\theta_{I(j)} + \zeta)\,((1-\lambda)T)^{1/2}\big) \\
& + \; e^{-\,0.5\theta^2_{I(j)}(1-\lambda)T} e^{0.5\theta^2_{I(j)}(1-\lambda)T} \\
& \times \big(1 - N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} \; + \; \theta_{I(j)}\,((1-\lambda)T)^{1/2}\big)\big) \\
& = \; e^{C_t + (\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta))T} \\
& \times N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} \; - \; (-\theta_{I(j)} + \zeta)\,((1-\lambda)T)^{1/2}\big) \\
& + \big(1 - N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} \; + \; \theta_{I(j)}\,((1-\lambda)T)^{1/2}\big)\big) \; .
\end{aligned}
\tag{60}
$$

The following lemma is well known.

Lemma E.1 *We have*

$$
\lim_{x \to -\infty} \frac{N(x)}{e^{-x^2/2}(-x)^{-1}} \; = \; \lim_{x \to +\infty} \frac{1 - N(x)}{e^{-x^2}x^{-1}} \; = \; \frac{1}{\sqrt{2\pi}}.
$$

Because, for generic parameter values, we will have exponential decay, the factor x^{-1} in the asymptotics of Lemma E.1 can be neglected. Similarly, by the strong law of large numbers for Brownian motion, $W_{\lambda T}/T \to 0$ almost surely, and therefore, in the expressions of the form $W_{\lambda T} + a\,T \; = \; T\,(a + W_{\lambda T}/T)$, the term with $W_{\lambda T}$ can be also ignored when calculating long run behavior.

We first observe that the term

$$
\begin{aligned}
& e^{(\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta))T} \\
& \times N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} \; - \; (-\theta_{I(j)} + \zeta)\,((1-\lambda)T)^{1/2}\big)
\end{aligned}
\tag{61}
$$

always converges to zero. Indeed, if $\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta) < 0$ then we *are done. If* $\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta) \; > \; 0$ *we have*

$$
\begin{aligned}
& -\zeta^{-1}\eta\, T\,((1-\lambda)T)^{-1/2} \;-\; (-\theta_{I(j)} + \zeta)\,((1-\lambda)T)^{1/2} \\
&= \; -\zeta^{-1}\,((1-\lambda)T)^{-1/2}\, T\,(\eta + (1-\lambda)\zeta(-\theta_{I(j)} + \zeta)) \\
&< \; -\zeta^{-1}\,((1-\lambda)T)^{-1/2}\, T\,(\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta))
\end{aligned}
\tag{62}
$$

is negative and converges to $-\infty$, *so that Lemma E.1 applies and we need to show that*

$$
\eta + (1-\lambda)\zeta(-\theta_{I(j)} + 0.5\zeta) \;-\; 0.5\zeta^{-2}\,(1-\lambda)^{-1}(\eta + (1-\lambda)\zeta(-\theta_{I(j)} + \zeta))^2 \;<\; 0\,,
$$

that is,

$$
\begin{aligned}
& \zeta^2\,\eta\,(1-\lambda) \;+\; (1-\lambda)^2\zeta^3\,(-\theta_{I(j)} + 0.5\zeta) \;-\; 0.5\eta^2 \\
& - 0.5(1-\lambda)^2\zeta^2(-\theta_{I(j)} + \zeta)^2 - \eta(1-\lambda)\zeta(-\theta_{I(j)} + \zeta) \\
&= \; - \, 0.5\eta^2 \, - 0.5(1-\lambda)^2\zeta^2(\theta_{I(j)}^2) + \eta(1-\lambda)\zeta\theta_{I(j)} \\
&= \; -0.5(\eta - (1-\lambda)\zeta\theta_{I(j)})^2 \;<\; 0
\end{aligned}
\tag{63}
$$

which is what had to be proved. Thus, we only need to show that the term

$$
1 - N\big(-\zeta^{-1}(\eta\, T + C_t)((1-\lambda)T)^{-1/2} \;+\; \theta_{I(j)}\,((1-\lambda)T)^{1/2}\big)
$$

converges to zero. This happens precisely when $0 \;<\; -\zeta^{-1}\eta \;+\; \theta_{I(j)}\,(1-\lambda)\,.$ *We have*

$$
\begin{aligned}
& -\zeta^{-1}\eta \;+\; \theta_{I(j)}\,(1-\lambda) \\
&= \; -(b_i\,(\theta_{I(j)} \;-\; \theta_i))^{-1}\, b_i(\kappa_{I(j)} - \kappa_i) \;+\; \theta_{I(j)}\,(1-\lambda)\,.
\end{aligned}
\tag{64}
$$

By the definition of the agent $I(j)$, *we have*

$$
\kappa_{I(j)} \;-\; (1-\lambda)\,0.5\,\theta_{I(j)}^2 = l_{I(j)} \;<\; l_i(\lambda) = \kappa_i \;-\; (1-\lambda)\,0.5\,\theta_i^2\,.
$$

Therefore,

$$
\begin{aligned}
\kappa_{I(j)} \;-\; \kappa_i \;&<\; (1-\lambda)\,0.5\,(\theta_{I(j)}^2 - \theta_i^2) \\
&= (1-\lambda)\,0.5\,(-(\theta_{I(j)} - \theta_i)^2 + 2\theta_{I(j)}(\theta_{I(j)} - \theta_i)) \\
&\le\; (1-\lambda)\,\theta_{I(j)}(\theta_{I(j)} - \theta_i)\,,
\end{aligned}
\tag{65}
$$

which is what had to be proved. The case $\zeta < 0$ *is completely analogous.*

Proposition E.1 *The drift* $\mu_t^{c_1/c_2}$ *of the log consumption ratio* $\log(c_{1t}/c_{2t})$ *such that*

$$E_t[d\log(c_{1t}/c_{2t})] \;=\; \mu_t^{c_1/c_2}\,dt$$

is given by

$$\begin{array}{ll} \ell_t^2 - \ell_t^1 & \textit{under the physical measure} \\ \hat{\ell}_t^2 - \hat{\ell}_t^1 & \textit{under the risk-neutral measure} \\ \tilde{\ell}_t^2 - \tilde{\ell}_t^1 & \textit{under the } T\textit{-forward measure} \end{array} \tag{66}$$

where

$$\begin{aligned} \ell_t^i &= b_i\,(\kappa_i - r_t - 0.5\,\theta_t^2) \\ \hat{\ell}_t^i &= \ell_t^i + b_i\,(\theta_t + \delta_i)\,\theta_t \\ \tilde{\ell}_t^i &= \ell_t^i + b_i\,(\theta_t + \delta_i)\,\hat{\theta}_t^T \end{aligned} \tag{67}$$

where

$$\hat{\theta}_t^T \;=\; E_t^{Q^T}[\theta_T]$$

is the market price of risk under the T*-forward measure.*

Proposition E.2 *For* $\lambda \in (\lambda_j, \lambda_{j+1})$*, we have*

$$E_{\lambda T}^{Q^T}[\theta_T] \;\to\; \theta_{I(j)}$$

almost surely under the physical measure.

Proof 2 *Indeed, it follows directly from* (51) *that*

$$\frac{\min_j b_j}{\max_j b_j}\,c_{iT}\,D_T^{-1} \;\le\; \omega_{iT} \;\le\; \frac{\max_j b_j}{\min_j b_j}\,c_{iT}\,D_T^{-1}\,,$$

and the required assertion follows directly from the identity.

$$\theta_T \;=\; \sum_i \omega_{iT}\,\theta_i\,.$$

Proof 3 (Proof of Proposition E.1) *We have*

$$c_{it} \;=\; e^{-\rho_i b_i t}\,M_t^{-b_i}\,Z_{it}^{b_i}\,c_{i0}\,,$$

and therefore

$$c_{1t}/c_{2t} \;=\; e^{(\rho_2 b_2 - \rho_1 b_1)t}\,M_t^{b_2 - b_1}\,e^{(b_1\delta_1 - b_2\delta_2)W_t + 0.5(b_2\delta_2^2 - b_1\delta_1^2)t}\,.$$

Recalling that

$$M_t^{-1}\,dM_t \;=\; -r_t dt \;-\; \theta_t\,dW_t\,,$$

we get that, under the physical measure,

$$\begin{aligned}
&(c_{1t}/c_{2t})^{-1}\,d(c_{1t}/c_{2t}) \;=\; ((\rho_2 b_2-\rho_1 b_1)+0.5(b_2\delta_2^2-b_1\delta_1^2))dt\\
&+\;(b_2-b_1)(M_t^{-1}dM_t+0.5(b_2-b_1-1)\theta_t^2 dt)\\
&+(b_1\delta_1-b_2\delta_2)dW_t+0.5(b_1\delta_1-b_2\delta_2)^2dt \;-\; (b_2-b_1)\theta_t(b_1\delta_1-b_2\delta_2)dt\\
&=\;\Big((\rho_2 b_2-\rho_1 b_1)+0.5(b_2\delta_2^2-b_1\delta_1^2)-(b_2-b_1)r_t+0.5(b_2-b_1)(b_2-b_1-1)\theta_t^2\\
&+0.5(b_1\delta_1-b_2\delta_2)^2 \;-\; (b_2-b_1)\theta_t(b_1\delta_1-b_2\delta_2)\Big)dt\\
&+\;(-(b_2-b_1)\theta_t+(b_1\delta_1-b_2\delta_2))dW_t
\end{aligned} \tag{68}$$

and the claim follows by direct calculation. The only exception is the case under the T-forward measure, which follows from Lemma E.2 below.

Proof 4 (Proof of Proposition 7.4) *The proof follows directly from (68) and Proposition 5.22, p.345 of Karatzas and Shreve (2008).*

Lemma E.2 *The density process of the T-forward measure is*

$$e^{-\int_0^t \hat\theta_t^T\,dW_t\,-\,0.5\,\int_0^t(\hat\theta_t^T)^2dt}\,,$$

where

$$\hat\theta_t^T \;=\; E_t^{Q^T}[\theta_T]\,.$$

Proof 5 (Proof of Lemma E.2) *A direct calculation, based on the same Malliavin calculus techniques implies that, under the risk-neutral measure,*

$$dB(t,T) \;=\; r_t dt \;+\; \sigma^B(t,T)dW_t^Q\,,$$

where

$$\sigma^B(t,T) \;=\; \theta_t \;-\; E_t^{Q^T}[\theta_T] \;=\; \theta_t \;-\; \frac{E_t[M_T\theta_T]}{E_t[M_T]}\,.$$

Therefore,

$$dE_t[M_T] \;=\; d\,(M_t\,B(t,T)) \;=\; B(t,T)dM_t+M_t dB(t,T) \;-\; \sigma^B(t,T)\,\theta_t\,dt\,,$$

and hence

$$\hat\theta^T \;=\; E_t^{Q^T}[\theta_T]\,.$$

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