Model independent finance, Vovk's outer measure, and insider trading

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joint work with
Beatrice Acciaio, Mathias Beiglböck, Alexander Cox, Nicolas Perkowski, and David Prömel

Plan of the talk

1. optimal SEP + optimal transport perspective [Beiglböck, Cox, H]

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3. $\Rightarrow$ super - replication [Beiglböck, Cox, H, Perkowski, Prömel]

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4. Example: Insider trading [Acciaio, Cox, H]

The optimal transport problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ and $c : X \times X \to \mathbb{R}$.

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(KP) $\inf_{q \in \mathrm{Cpl}(\mu_0, \mu_1)} \int c(x, y) \; q(dx, dy)$

The optimal Skorokhod embedding problem (SEP)

Given $\mu \in \mathcal{P}(\mathbb{R})$, centered, find a stopping time τ s.t.

$$B_\tau \sim \mu, \quad B_{\cdot\wedge\tau} \text{ is u.i.}$$

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and which maximizes for a given functional γ

$$\mathbb{E}[\gamma((B_s)_{s\leq\tau}, \tau)].$$

The SEP as a transport problem

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$$\bar{\tau}(d\omega, dt) := \delta_{\tau(\omega)}(dt)\, \mathbb{W}(d\omega)$$

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The optimal SEP becomes:

$$P_{\mathsf{SEP}} = \sup_{\bar{\tau} \in \mathsf{RST}(\mu)} \int \gamma((\omega_s)_{s \le t}, t)\ \bar{\tau}(d\omega, dt)$$

which is a linear optimization problem!

The SEP as a transport problem

Theorem (BCH '14)

Setting

$$D_{\mathrm{SEP}} := \inf \left\{ a : \begin{array}{l} \textit{there exist } \psi, \int \psi \, d\mu = 0, \textit{ nice martingale } M, M_0 = 0, \\ a + \psi(\omega(t)) + M_t(\omega) \geq \gamma(\omega, t), \textit{ for all } \omega, t \end{array} \right\}$$

it holds that $P_{\mathrm{SEP}} = D_{\mathrm{SEP}}$.

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- $\Omega = C[0,\infty), \omega \in \Omega, B_t(\omega) = \omega_t$
- Simple strategy:

$$H_t(\omega) = \sum_{n \geq 0} F_n(\omega) 1_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t)$$

with stopping times $\tau_0 < \tau_1 < \tau_2 < \dots$ satisfying $\lim_n \tau_n(\omega) \nearrow \infty$ for all ω and $F_n \in m\mathcal{F}_{\tau_n}$.

→ $(H \cdot B)_t(\omega)$ well defined for all ω, t

- For $\lambda > 0$ the set of λ- admissable strategies is defined as

$$\mathcal{H}_\lambda = \{H \ : \ (H \cdot B)_t \geq -\lambda, \text{ for all } t, \omega\}.$$

Vovk's outer measure $\bar{\mathcal{P}}$

Definition

a) For $A \subset \Omega$ the outer measure $\bar{\mathcal{P}}$ is defined by

$$\bar{\mathcal{P}}(A) := \inf\{\lambda : \text{ there exists } (H_n)_n \subset \mathcal{H}_\lambda \text{ s.t.} \\ \liminf_{t\to\infty} \liminf_{n\to\infty} \lambda + (H_n \cdot B)_t(\omega) \geq 1_A(\omega) \text{ for all } \omega\}.$$

b) $A \subset \Omega$ is called *null* if $\bar{\mathcal{P}}(A) = 0$.

c) A property (P) holds for *typical price paths* if

$$\bar{\mathcal{P}}((P) \text{ is violated}) = 0.$$

Properties of $\bar{\mathcal{P}}$

Proposition

a) $\bar{\mathcal{P}}(A) = 0$ *iff there exists* $(H_n)_n \subset \mathcal{H}_1$ *such that*

$$1 + \liminf_{t\to\infty} \liminf_{n\to\infty} (H \cdot B)_t(\omega) \geq \infty \cdot 1_A(\omega) \qquad \textit{for all } \omega.$$

b) *Let Q be a probability measure on $(\Omega, \mathcal{F})$ such that B is a local martingale, $A \in \mathcal{F}$, then $Q(A) \leq \bar{P}(A)$.*

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Theorem (Vovk)

For typical price paths there exists a quadratic variation process, denoted by $\langle B \rangle_t(\omega)$.

Vovk's very remarkable DDS Theorem

Set

$$\tau_t(\omega) := \inf\{s \geq 0 \ : \ \langle B \rangle_s(\omega) \geq t\}$$

and define maps $\mathrm{ntt} : \Omega \to \Omega$ and $\bar{\mathrm{ntt}} : \Omega \times \mathbb{R}_+ \to \Omega \times \mathbb{R}_+$ via

$$(\mathrm{ntt}(\omega))_t := \omega_{\tau_t(\omega)}, \quad \bar{\mathrm{ntt}}(\omega, t) := (\mathrm{ntt}(\omega), \langle B \rangle_t(\omega)).$$

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Theorem (Vovk)

Let F be bounded, Borel, and non-negative. Then

$$\bar{\mathbb{E}}[F \circ \mathrm{ntt}, \langle B \rangle_\infty = \infty] = \int F \ \mathbb{W}(d\omega).$$

Duality on $C[0,1]$ for time invariant derivatives

Interested in

$$\omega \mapsto G((S_t)_{t\le 1}(\omega)) = G((S_t)_{t\le 1}(\omega), \langle S\rangle_1(\omega)) := \tilde{G} \circ \mathrm{n\bar{t}t}(\omega, 1)$$

for

$$\tilde{G}(\omega, t) = \gamma(\omega, t),$$

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i.e. we are interested in

$$P_M := \sup_{\mathbb{P}\text{: loc. mg. meas.},(S_1)_*\mathbb{P}\sim\mu} \mathbb{E}_{\mathbb{P}}[G((S_t)_{t\leq 1})] \overset{?}{=} D_M$$

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Task: Find the trading strategy corresponding to the martingale in the SEP duality

Trading via Vovk's DDS Theorem

For all $a > P_{\mathsf{SEP}}$ there are $\psi, \int \psi d\mu = 0$ and a martingale $M, M_0 = 0$ s.t. *for all* ω, t

$$\gamma(\omega, t) - a - \psi(\omega(t)) \leq M_t(\omega).$$

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In particular, it holds *for all* ω, t

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In particular, this holds for $t = 1$ yielding the desired duality result.

A (prototypical) n-marginal duality result

Theorem (BCHPP '15)

Let $I \subset \{1, \ldots, n\}, n \in I, (\mu_j)_j \in I$ *be increasing in convex order and*

$$G(\omega) = \gamma(\mathrm{ntt}(\omega)_{\restriction[0,\langle\omega\rangle_n]}, \langle\omega\rangle_1, \ldots, \langle\omega\rangle_n).$$

Set

$$P_n = \sup\{\mathbb{E}_{\mathbb{P}}[G] \ : \ \mathbb{P} \textit{ loc. mg. meas. on } C[0,n], S_0 = 0, S_j \sim \mu_j \ \forall\, j \in I\}$$

and

$$D_n = \inf\Big\{a \ : \ \exists H, \psi_j, \int \psi_j d\mu_j = 0 \ \forall j \in I \textit{ s.t.}$$
$$a + \sum_{j\in I} \psi_j(S_j(\omega)) + (H \cdot S)_n \geq G((S_t)_{t\leq n}(\omega))\Big\}.$$

Then, there is no duality gap, i.e. $P_n = D_n$.

Insider trading

Within this framework we can model insider information, yielding:

- very general dual theory
- connection of robust arbitrage and specific properties of solutions to SEP, i.p. we can characterise robust arbitrage via geometric properties of the optimal solution to the corresponding SEP
- in certain cases: explicit optimal strategies

Thanks for your attention