# Affine realizations with affine state processes for the HJMM equation

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# Introduction

- Consider a stochastic partial differential equation (SPDE)

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 = h_0. \end{cases} \quad (1)$$

- Affine realization: For every starting point $h_0$ we have

$$r = \psi + X \quad \text{(locally)}$$

where $\psi : \mathbb{R}_+ \to H$ and $X \in V$ with $\dim V < \infty$.
- Affine state process: $X$ is an affine process.
- Goal of this talk: Characterization result in terms of $(A, \alpha, \sigma)$.
- Particular example: HJMM equation for interest rates.

# Contribution to the literature

- Affine realizations (among others):
  - Björk & Svensson (Math. Finance 2001)
  - Filipović & Teichmann (J. Funct. Anal. 2003)
  - Tappe (Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 2010)
- Affine processes in finance (among others):
  - Duffie & Kan (Math. Finance 1996)
  - Duffie, Filipović & Schachermayer (Ann. Appl. Probab. 2003)
  - Filipović (Stochastic Process. Appl. 2005)
- In this talk, we investigate when the state processes of an affine realization are affine processes.

# General framework

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ is a stochastic basis.
- $W$ is a $\mathbb{R}^n$-valued Wiener process.
- $H$ is a separable Hilbert space.
- Let $\alpha : H \to H$ and $\sigma : H \to H^n$ be continuous.
- $A : \mathcal{D}(A) \subset H \to H$ generates a $C_0$-semigroup $(S_t)_{t \geq 0}$ on $H$.
- That is, a family $S_t \in L(H)$, $t \geq 0$ such that:
    - $S_0 = \mathrm{Id}$;
    - $S_{s+t} = S_s S_t$ for all $s, t \geq 0$;
    - $\lim_{t \to 0} S_t h = h$ for all $h \in H$.
- Furthermore, we have

$$Ah = \lim_{t \to 0} \frac{S_t h - h}{t} \quad \text{for all } h \in \mathcal{D}(A).$$

# Solution concepts

- Strong solution: We have $r \in \mathcal{D}(A)$ and

$$r_t = h_0 + \int_0^t (Ar_s + \alpha(r_s))ds + \int_0^t \sigma(r_s)dW_s.$$

- Weak solution: For each $\zeta \in \mathcal{D}(A^*)$ we have

$$\langle \zeta, r_t \rangle = \langle \zeta, h_0 \rangle + \int_0^t \big( \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle \big) ds + \int_0^t \langle \zeta, \sigma(r_s) \rangle dW_s.$$

- Mild solution: We have

$$r_t = S_t h_0 + \int_0^t S_{t-s} \alpha(r_s) ds + \int_0^t S_{t-s} \sigma(r_s) dW_s.$$

- Strong solution $\Rightarrow$ Weak solution $\Rightarrow$ Mild solution.

# HJMM equation for interest rates

- Heath-Jarrow-Morton (HJM) forward rate model

$$f_t(T) = f_0(T) + \int_0^t \alpha_s(T)ds + \int_0^t \sigma_s(T)dW_s, \quad t \in [0, T]$$

- with Musiela parametrization

$$r_t(x) = f_t(t + x), \quad x \in \mathbb{R}_+.$$

- HJMM equation on a space $H$ of functions $h : \mathbb{R}_+ \to \mathbb{R}$.
- We have $A = d/dx$ and the HJM drift condition

$$\alpha_{\mathrm{HJM}}(h) = \sum_{k=1}^n \sigma_k(h) \int_0^\bullet \sigma_k(\eta)d\eta.$$

# Invariant foliations

- Recall the SPDE (1), which is given by

$$\begin{cases} dr_t & = & (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 & = & h_0. \end{cases}$$

- Let $h_0 \in H$ be a starting point, and suppose that

$$r = \psi + X,$$

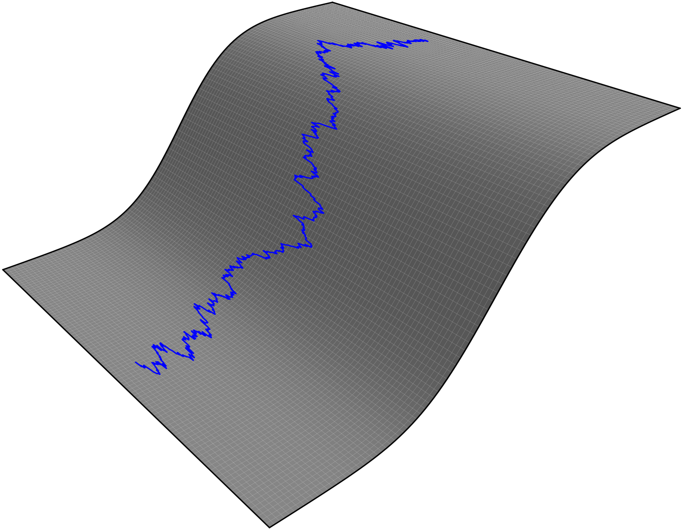
where $\psi : \mathbb{R}_+ \to H$ and $X \in V$ with $\dim V < \infty$.

- Then the foliation $(\mathcal{M}_t)_{t \in \mathbb{R}_+}$ given by

$$\mathcal{M}_t = \psi(t) + V, \quad t \in \mathbb{R}_+,$$

is invariant for the SPDE (1).

# Path of the solution process on an invariant foliation



# Tangential conditions

- A foliation $(\mathcal{M}_t)_{t \in \mathbb{R}_+}$ is invariant for (1) if and only if

$$\begin{aligned} \mathbb{M} &\subset \mathcal{D}(A), \\ \beta(h) &\in T\mathcal{M}_t, \quad t \in \mathbb{R}_+ \text{ and } h \in \mathcal{M}_t, \\ \sigma(\mathbb{M}) &\subset V^n. \end{aligned}$$

- Here we use the notations

$$\mathbb{M} := \bigcup_{t \in \mathbb{R}_+} \mathcal{M}_t, \quad \beta := A + \alpha, \quad T\mathcal{M}_t := \frac{d}{dt}\psi(t) + V.$$

- We would like that $X$ is affine; that is $X \in \mathfrak{C} \oplus U \subset V$ and

$$\mathbb{E}[e^{\langle u, X_T \rangle} \mid \mathcal{F}_t] = \exp(\phi(t, T, u) + \langle \psi(t, T, u), X_t \rangle).$$

- Here $\mathfrak{C} \cong \mathbb{R}^d_+$ is a proper cone, and $U \cong \mathbb{R}^{m-d}$ is a subspace.

# Criteria for affine state processes

- Letting $H = G \oplus V$, we have $\mathbb{M} = \partial\mathbb{M} \oplus \mathfrak{C} \oplus U$, and require:
  1. For each $g \in \partial\mathbb{M}$ the mapping
  $$v \mapsto \Pi_V \beta(g+v) : \mathfrak{C} \oplus U \to V$$
  is affine and inward pointing at boundary points.
  2. For each $g \in \partial\mathbb{M}$ the mapping
  $$v \mapsto \sigma(g+v) : \mathfrak{C} \oplus U \to V^n$$
  is square-affine and parallel at boundary points.
- Using the identification $V^n \cong L(\mathbb{R}^n, V)$, we consider
$$v \mapsto \sigma^2(g+v) := \sigma(g+v)\sigma^*(g+v) : \mathfrak{C} \oplus U \to L(V).$$

# Affine realizations with affine state processes

- Recall the SPDE (1), which is given by

$$\begin{cases} dr_t & = & (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 & = & h_0. \end{cases}$$

- Let $\mathfrak{I} \subset H$ be a set of initial points.
- Assumptions: $\mathfrak{I} = \partial\mathfrak{I} \oplus \mathfrak{C} \oplus U$ and $H = \overline{\langle \partial\mathfrak{I} \rangle} \oplus V$.
- Criterion in terms of $(A, \alpha, \sigma)$ and $\mathfrak{I}$: We have $\mathfrak{I} \subset \mathcal{D}(A)$ and $\sigma(\mathfrak{I}) \subset V^n$, and for each $g \in \partial\mathfrak{I}$ we have

$$\beta_g(v) \in V, \quad v \in \mathfrak{C} \oplus U,$$

$$\Pi_V \beta(g + \cdot) \text{ is affine and inward pointing},$$

$$\sigma(g + \cdot) \text{ is square-affine and parallel},$$

where $\beta = A + \alpha$ and $\beta_g(v) = \beta(g + v) - \beta(g)$.

# Particular structure of the drift

- Assume that $\alpha = S\sigma^2$ with $S \in L(L(V), H)$.
- This is the case for the HJMM equation for interest rates.
- If $\sigma(g + \cdot)$ is square-affine, then $\Pi_V \beta(g + \cdot)$ is affine.
- If $\sigma(g + \cdot)$ is parallel, then $\Pi_V \beta(g + \cdot)$ does not need to be inward pointing.
- Criterion in terms of $(A, \sigma)$ and $\mathfrak{I}$: For each $g \in \partial\mathfrak{I}$ we have

$$\begin{aligned} &\sigma(g + \cdot) \text{ is square-affine and parallel,} \\ &\Pi_V(Ag + S\sigma^2(g)) \in \mathfrak{C} \oplus U, \\ &Ac + S\sigma_g^2(c) \in (\mathfrak{C} + \langle c \rangle) \oplus U, \quad c \in \partial\mathfrak{C}, \\ &Au \in U, \quad u \in U. \end{aligned}$$

- Under suitable conditions on $S$, the state processes of an affine realization have affine characteristics.

# The HJMM equation

- Consider the HJMM (Heath-Jarrow-Morton-Musiela) equation

$$\begin{cases} dr_t = \left(\frac{d}{dx} r_t + \alpha_{\mathrm{HJM}}(r_t)\right) dt + \sigma(r_t) dW_t \\ r_0 = h_0. \end{cases} \tag{2}$$

- We fix a nondecreasing $C^1$-function $w : \mathbb{R}_+ \to [1, \infty)$ such that $w^{-1/3} \in \mathcal{L}^1(\mathbb{R}_+)$, and denote by $H$ the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_H := \left( |h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx \right)^{1/2} < \infty.$$

- To ensure the absence of arbitrage, the drift is given by

$$\alpha_{\mathrm{HJM}}(h) = \sum_{k=1}^n \sigma_k(h) \int_0^\bullet \sigma_k(\eta) d\eta.$$

# The Vasiček model

- Let $U \subset H$ be a subspace with $\dim U = 1$.
- Suppose that $\sigma(h) = \Phi(h)\lambda$ with $\Phi : H \to \mathbb{R}$ and $\lambda \in U$.
- The following statements are equivalent:
  1. The HJMM equation (2) has an affine realization generated by $U$ with initial curves $\mathcal{D}(d/dx)$.
  2. The HJMM equation (2) has an affine realization with affine state processes generated by $U$ with initial curves $\mathcal{D}(d/dx)$.
  3. There are constants $\rho, \gamma \in \mathbb{R}$ such that

     $$\lambda(x) = \rho \cdot e^{-\gamma x}, \quad x \in \mathbb{R}_+,$$

     and for each $h \in H$ the mapping $\Phi(h + \cdot)$ is constant.
- Note that the state processes are $U$-valued, and that $U \cong \mathbb{R}$.

# The Cox-Ingersoll-Ross model

- Let $\mathfrak{C} \subset H$ be a proper cone with $\dim \mathfrak{C} = 1$, and let $\mathfrak{I} \subset H$.
- Suppose that $\sigma(h) = \Phi(h)\lambda$ with $\Phi : H \to \mathbb{R}$ and $\lambda \in \mathfrak{C}$.
- Suppose that the HJMM equation (2) has an affine realization generated by $\mathfrak{C}$ with initial curves $\mathfrak{I}$.
- Then there are $\Psi : H \to \mathbb{R}$ and $\ell \in H^*$ with $\ell(\lambda) = 1$ and $\rho > 0$, $\gamma \in \mathbb{R}$ such that

$$\Phi(h) = \rho\sqrt{\Psi(\Pi_G h) + \ell(h)}, \quad h \in \mathfrak{I},$$
$$\frac{d}{dx}\lambda + \rho^2 \lambda \Lambda + \gamma\lambda = 0.$$

- We have affine state processes if and only if $\Psi \equiv 0$, that is

$$\Phi(h) = \rho\sqrt{\ell(h)}, \quad h \in \mathfrak{I}.$$

# The set of initial curves

- We assume that $\sigma(h) = \rho\sqrt{|\ell(h)|}\lambda$ for $h \in H$.
- Here we have $\rho > 0$ and $\ell \in H^*$ with $\ell(\lambda) = 1$.
- For some $\gamma \in \mathbb{R}$ the function $\lambda$ is a solution of

$$\frac{d}{dx}\lambda + \rho^2 \lambda \Lambda + \gamma \lambda = 0.$$

- The HJMM equation (2) has an affine realization with affine state processes generated by $\langle \lambda \rangle^+$ and initial curves

$$\mathfrak{I} = \{h \in \mathcal{D}(d/dx) : \ell(h) \geq 0 \text{ and } \ell(h') + (\rho^2 \ell(\lambda\Lambda) + \gamma)\ell(h) > 0\}.$$

# Further consequences

- Suppose that $\ell(h) = h(0)$. Then we have

$$\mathfrak{I} = \{h \in \mathcal{D}(d/dx) : h(0) \geq 0 \text{ and } h'(0) + \gamma h(0) > 0\}.$$

- For $h \in \mathfrak{I}$ we can choose $r(0) \in \mathbb{R}_+$ as state process.
- Therefore, we have $\mathbb{P}(r_t(0) \geq 0) = 1$ for all $t \in [0, \delta]$.
- The expectation hypothesis implies

$$h(t) = \mathbb{E}_{\mathbb{P}^t}[r_t(0)] \geq 0 \quad \text{for all } t \in [0, \delta].$$

- Indeed, if not $h(0) > 0$, then $h(0) = 0$ and $h'(0) > 0$.

# Another example

- Suppose that $\sigma(h) = \rho\sqrt{|\ell(h)|}\lambda$ for $h \in H$.
- Here $\rho > 0$ and $\lambda(x) = e^{-\gamma x}$ for $x \in \mathbb{R}_+$.
- The functional $\ell \in H^*$ satisfies $\ell(\lambda) = 1$ and $\ell(\lambda^2) = 0$.
- The HJMM equation (2) has an affine realization generated by $\langle \lambda \rangle$, but the state processes are not affine.
- However, the HJMM equation (2) has an affine realization with affine state processes generated by $\langle \lambda \rangle^+ \oplus \langle \lambda^2 \rangle$ and initial curves

$$\mathfrak{I} = \{ h \in \mathcal{D}(d/dx) : \ell(h) \geq 0 \text{ and } \ell(h' + \gamma \cdot h) > 0 \}.$$

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