A Bayesian methodology for systemic risk assessment in financial networks

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The problem

- Consider interbank market as network:
  - Nodes consist of n banks with indices in $\mathcal{N} = \{1, \ldots, n\}$.
  - Edges L_{ij} represent nominal interbank liability of bank i to bank j.
- Stress tests: Suppose some banks default on their liabilities. How do losses spread along the edges? What if edges are not observable?
- A matrix $L = (L_{ij}) \in \mathbb{R}^{n \times n}$ is a liabilities matrix if $L_{ij} \geq 0$, $L_{ii} = 0 \; \forall i, j$
- Total nominal interbank liabilities of bank i: $r_i(L) := \sum_{j=1}^{m} L_{ij}$.
- Total nominal interbank assets of bank i: $c_i(L) := \sum_{j=1}^{m} L_{ji}$.
- In practice, L_{ij} not fully observable, but $r_i(L)$, $c_i(L)$ are.
- How to fill in the missing data? Implications for stress testing?

Main contributions

- Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.
- Application to systemic risk assessment:
  - Can give probabilities for outcomes of stress tests.
  - Results show limitations of classical approach.
- Show that general monotonicity arguments relating severity of systemic risk to number of edges do not hold in general.
- Code is available as R-package (systemicrisk) on CRAN.

Existence of admissible liabilities matrix

Theorem (Existence of an admissible liabilities matrix)

Consider two vectors $a, l \in [0, \infty)^n$ *satisfying* $\sum_{i=1}^n a_i = \sum_{i=1}^n l_i$. *Then there exists a matrix* $L \in [0, \infty)^{n \times n}$ *with*

$$\operatorname{diag}(L) = 0, \quad c(L) = a, \quad r(L) = l$$

if and only if

$$a_i \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} l_j \quad \forall i \in \mathcal{N}.$$

Proof contains algorithm giving explicit construction .

The Bayesian framework

- Constructs adjacency matrix $\mathcal{A} = (\mathcal{A}_{ij})$; attaches liabilities L_{ij}.
- Model:

$$
\begin{aligned}
&\text{For } i, j \in \mathcal{N}: \\
&\qquad \mathcal{A}_{ij} \sim \text{Bernoulli}(p_{ij}) \\
&\qquad L_{ij} | \{\mathcal{A}_{ij} = 1\} \sim \text{Exponential}(\lambda_{ij}) \\
&\qquad L_{ij} = 0 \text{ if } \mathcal{A}_{ij} = 0.
\end{aligned}
$$

- Parameters:
    - $p \in [0, 1]^{n \times n}$, $\text{diag}(p) = 0$; p_{ij} probability of existence of directed edge from i to j,
    - $\lambda \in \mathbb{R}^{n \times n}$, governs distribution of weights given that edge exists.

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- Observations: $c(L) = a \in \mathbb{R}^n$, $r(L) = l \in \mathbb{R}^n$.

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- Observations: $c(L) = a \in \mathbb{R}^n$, $r(L) = l \in \mathbb{R}^n$.
- Main interest: Distribution of $h(L) \mid a, l$.

Gibbs sampling for $L|a, l$

- Markov Chain Monte Carlo (MCMC): Interested in sampling from a given distribution. Construct a Markov chain with this stationary distribution. Run chain. Chain converges to stationary distribution.
- Key idea of Gibbs sampler: a step of the chain updates one or several components of the entire parameter vector by sampling them from their joint conditional distribution given the remainder of the parameter vector.
- Here parameter vector is matrix L:
  - Initialise chain with matrix L that satisfies $r(L) = l$, $c(L) = a$.
  - MCMC sampler produce a sequence of matrices $L^1, L^2, \dots$.
  - Quantity of interest: $\mathbb{E}[h(L)|l, a] \approx \frac{1}{N}\sum_{i=1}^{N} h(L^{i\delta+b})$,

    N number of samples, b burn-in period, $\delta \in \mathbb{N}$ thinning parameter.

Updating components of L

- Need to decide which elements of L need to be updated.
- Need to determine how the new values will be chosen, i.e., need to determine their distribution conditional on remainder of elements of L.

Illustration of updating submatrices

$L_{i_1 j_1}$	$L_{i_1 j_2}$
$L_{i_2 j_1}$	$L_{i_2 j_2}$

	$L_{i_1 j_1}$	$L_{i_1 j_2}$
$L_{i_2 j_3}$		$L_{i_2 j_2}$
$L_{i_3 j_3}$	$L_{i_3 j_1}$	

	$L_{i_1 j_1}$	$L_{i_1 j_2}$	
		$L_{i_2 j_2}$	$L_{i_2 j_3}$
$L_{i_3 j_4}$			$L_{i_3 j_3}$
$L_{i_4 j_4}$	$L_{i_4 j_1}$		

Updating - Illustration

47
49.9
91.3
99.9
66.5
54.9

47.1	49.8	91.2	100	66.5	54.9

→

	0	14.7	32.3	0	0	47
0		19.5	11	0	19.5	49.9
23.5	20.3		20.5	20.5	6.4	91.3
23.5	20.3	23.5		23	9.5	99.9
0	9.1	19.5	18.4		19.5	66.5
0	0	14	17.9	23		54.9

47.1	49.8	91.2	100	66.5	54.9

→

	0	13.9	32.3	0	0.8	47
0		19.5	11	0	19.5	49.9
23.5	20.3		20.5	20.5	6.4	91.3
23.5	20.3	23.5		23	9.5	99.9
0	9.1	20.3	18.4		18.6	66.5
0	0	14	17.9	23		54.9

47.1	49.8	91.2	100	66.5	54.9

→

	0	13.9	32.3	0	0.8	47
0		35.5	11	0	3.5	49.9
23.5	20.3		4.5	20.5	22.5	91.3
23.5	20.3	23.5		23	9.5	99.9
0	9.1	4.3	34.4		18.6	66.5
0	0	14	17.9	23		54.9

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0		35.5	11	0	3.5	49.9
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0	9.1	4.3	34.4		18.6	66.5
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Balance sheets and fundamental defaults

- Balance sheet of bank i:

Assets		Liabilities	
external assets interbank assets	$a_i^{(e)}$ $a_i := c_i(L)$	external liabilities interbank liabilities net worth	$l_i^{(e)}$ $l_i := r_i(L)$ $w_i := w_i(L, a_i^{(e)}, l_i^{(e)})$

$$w_i := w_i(L, a_i^{(e)}, l_i^{(e)}) := a_i^{(e)} + c_i(L) - l_i^{(e)} - r_i(L).$$

- Stress tests: apply proportional shock $s \in [0,1]^n$ to external assets; shocked external assets are $s_i a_i^{(e)}\ \forall i$.
- Fundamental defaults: $\{i \mid w_i(L, s_i a_i^{(e)}, l_i^{(e)}) < 0\}$
- Fundamental defaults can be checked from balance sheet aggregates without needing to know the whole matrix L!
- To check for contagious defaults we need to know L.

Empirical example - data

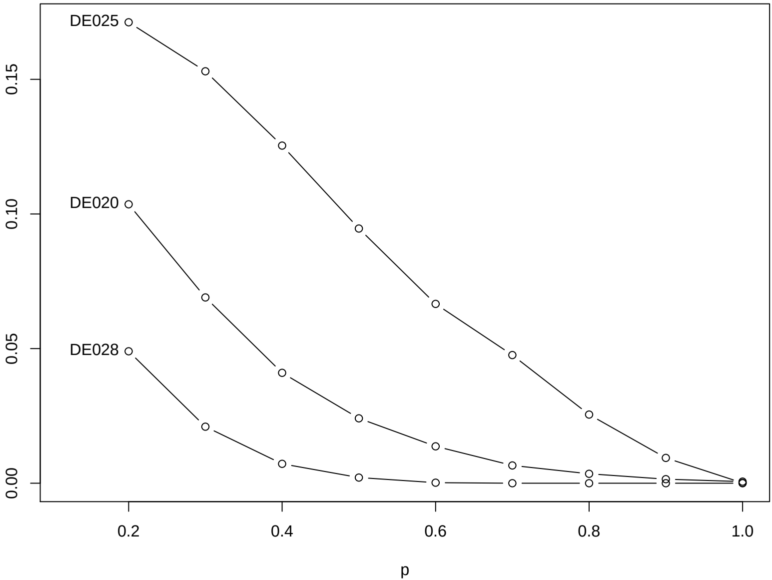
Balance sheet data (in million Euros) from banks in the EBA 2011 stress test:

Bank code	Bank	$a^{(e)} + a$	a	w
DE017	DEUTSCHE BANK AG	1,905,630	47,102	30,361
DE018	COMMERZBANK AG	771,201	49,871	26,728
DE019	LANDESBANK BADEN-WURTTEMBERG	374,413	91,201	9,838
DE020	DZ BANK AG	323,578	100,099	7,299
DE021	BAYERISCHE LANDESBANK	316,354	66,535	11,501
DE022	NORDDEUTSCHE LANDESBANK -GZ-	228,586	54,921	3,974
DE023	HYPO REAL ESTATE HOLDING AG	328,119	7,956	5,539
DE024	WESTLB AG, DUSSELDORF	191,523	24,007	4,218
DE025	HSH NORDBANK AG, HAMBURG	150,930	4,645	4,434
DE027	LANDESBANK BERLIN AG	133,861	27,707	5,162
DE028	DEKABANK DEUTSCHE GIROZENTRALE	130,304	30,937	3,359

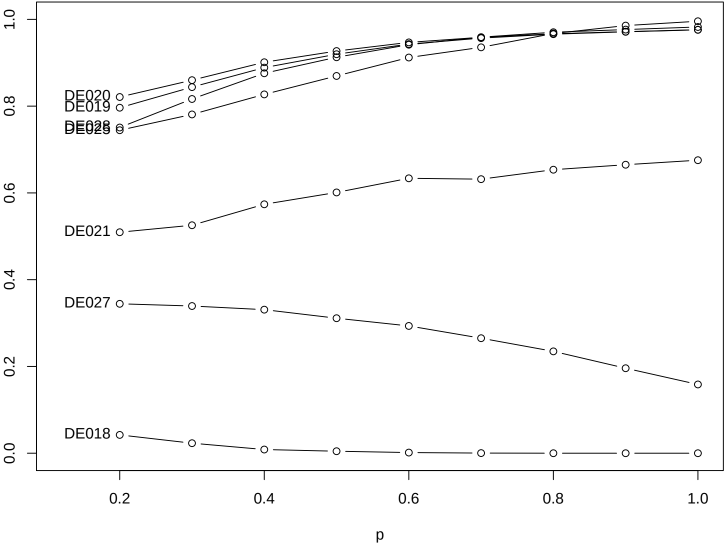
Stress testing

- We apply a deterministic shock to external assets of all 11 banks in the network by considering the shocked external assets $s_i a_i^{(e)}$ with $s_i = 0.97 \, \forall i \in \mathcal{N}$.
- Shock causes fundamental default of 4 banks: DE017, DE022, DE023, DE024.
- We apply the clearing approach by Eisenberg & Noe (2001) and [Rogers & V. (2013)] to determine which banks suffer contagious defaults.
- Gibbs sampler allows us to derive posteriori default probabilities for remaining 7 banks.

Default probabilities of banks as a function of p



Default probabilities for clearing with default costs



Mean out-degree of banks, i.e., $\mathbb{E}[\sum_j \mathcal{A}_{ij} \mid a, l]$, for different p^{ER} in the Erdős-Rényi network

	l	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
DE020	99936	3.50	4.40	5.40	6.20	6.90	7.60	8.30	9.00	10.00
DE019	91314	3.30	4.20	5.10	6.00	6.70	7.50	8.20	8.90	10.00
DE021	66494	2.90	3.70	4.70	5.50	6.40	7.20	8.00	8.80	10.00
DE022	54907	2.70	3.50	4.40	5.30	6.10	7.00	7.80	8.80	10.00
DE018	49864	2.60	3.40	4.30	5.10	6.00	6.90	7.80	8.70	10.00
DE017	46989	2.50	3.30	4.20	5.10	5.90	6.80	7.70	8.70	10.00
DE028	30963	2.20	2.80	3.60	4.50	5.40	6.30	7.30	8.40	10.00
DE027	27679	2.10	2.70	3.50	4.30	5.20	6.10	7.10	8.30	10.00
DE024	23971	1.90	2.60	3.30	4.10	5.00	5.90	7.00	8.20	10.00
DE023	8023	1.40	1.80	2.30	2.80	3.50	4.30	5.40	6.90	10.00
DE025	4841	1.20	1.50	1.90	2.40	2.90	3.60	4.60	6.10	10.00

Summary

- Development of Bayesian framework (Gibbs sampler) for sampling from distribution of liabilities matrix conditional on its row and column sums.
- Can be used for stress tests using empirical data.
- Can be used as a simulation tool to analyse heterogeneous networks.
- Can incorporate additional information such as expert views etc. on the network structure.
- R package (systemicrisk) available from CRAN.

References I

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