PROBLEM 1. (a)

$$m_i = \mathbb{E}\left[\sum_j b_j Y_j \middle| H = i\right] = \sum_j b_j \mathbb{E}\left[Y_j | H = i\right] = s_i \sum_j a_j b_j \mathbb{E}\left[Z_j\right] = s_i \sum_j a_j b_j, \quad (1)$$

$$v_i = \mathrm{Var}\left[\sum_j b_j Y_j \middle| H = i\right] \overset{(\star)}{=} \sum_j b_j^2 \,\mathrm{Var}\left[Y_j | H = i\right] = s_i^2 \sum_j a_j^2 b_j^2 \,\mathrm{Var}\left[Z_j\right] = s_i^2 \sum_j a_j^2 b_j^2, \quad (2)$$

where in $(\star)$ we use the fact that the Y_j are uncorrelated. Combining eqs. (1) and (2) gives

$$q = \frac{(m_0 - m_1)^2}{v_0 + v_1} = \frac{(s_0 - s_1)^2}{s_0^2 + s_1^2} \frac{\left(\sum_j a_j b_j\right)^2}{\sum_j a_j^2 b_j^2}. \quad (3)$$

(b) Let $\mathbf{c} = [a_1 b_1, \ldots, a_n b_n] \in \mathbb{R}^n$.

$$\left(\sum_j a_j b_j\right)^2 = \left(\sum_j c_j\right)^2 = |\langle \mathbf{c}, \mathbf{1}\rangle|^2 \overset{(\star)}{\leq} \|\mathbf{c}\|_2^2 \, \|\mathbf{1}\|_2^2 = n \sum_j a_j^2 b_j^2,$$

where $(\star)$ is due to the Cauchy-Schwarz inequality. Plugging the above into eq. (3) gives

$$q \leq n \frac{(s_0 - s_1)^2}{(s_0^2 + s_1^2)}. \quad (4)$$

(c) Quality q is maximized when eq. (4) holds with equality, which is true if and only if $\mathbf{c}$ and $\mathbf{1}$ are colinear, i.e.,

$$b_j = \frac{\lambda}{a_j}, \quad (5)$$

for some constant $\lambda \in \mathbb{R}$.

(d)

$$f_{Y|H}(y|i) = \prod_{j=1}^n f_{Y_j|H}(y_j|i) = \prod_{j=1}^n f_{Z_j}\left(\frac{y_j}{s_i a_j}\right) = \exp\left(-\frac{1}{s_i}\sum_{j=1}^n \frac{y_j}{a_j}\right) \overset{eq.~(5)}{=} \exp\left(-\frac{T}{\lambda s_i}\right).$$

As the joint distribution only depends on $Y = [Y_1, \ldots, Y_n]$ through T, the statistic is sufficient.

Problem 2. (a) A simple orthonormal basis for $\mathcal{W} = \{w_0, w_1, w_2, w_3\}$ is

$$\mathcal{B} = \{\psi_0(t) = \psi(t), \psi_1(t) = \psi(t-1)\}, \qquad \psi(t) = \mathbb{1}_{[0,1[}(t).$$

Using $\mathcal{B}$, the MAP optimal decision rule is given by

$$\hat{H} = \underset{i\in\{0,1,2,3\}}{\arg\max} \langle Y, c_i\rangle - \frac{1}{2}\|c_i\|_2^2,$$

where $Y = [\langle R, \psi_0\rangle, \langle R, \psi_1\rangle] \in \mathbb{R}^2$ and $c_i = [\langle w_i, \psi_0\rangle, \langle w_i, \psi_1\rangle] \in \mathbb{R}^2$. In practice we cannot obtain Y as above since we don't have a matched filter for ψ. Notice however that $\psi(t) = h_0(t) + h_1(t)$ such that

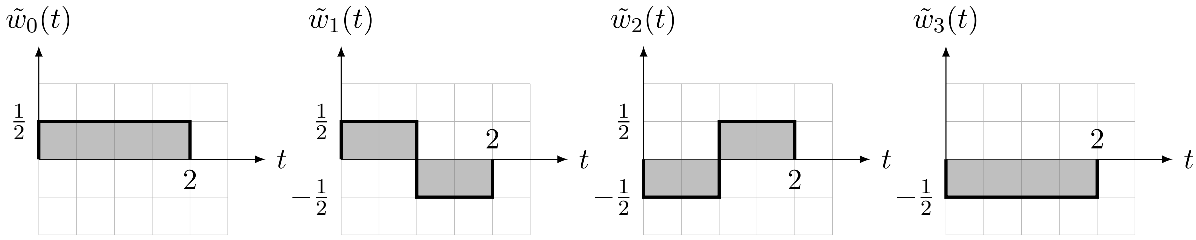
$$\begin{aligned}
\langle R, \psi_0\rangle &= \int R(t)\psi_0(t)\,dt = \int R(t)h_0(t)\,dt + \int R(t)h_1(t)\,dt \\
&= (R * h_1)(1) + (R * h_0)(1), \\
\langle R, \psi_1\rangle &= \int R(t)\psi_0(t-1)\,dt = \int R(t)h_0(t-1)\,dt + \int R(t)h_1(t-1)\,dt \\
&= (R * h_1)(2) + (R * h_0)(2).
\end{aligned}$$

Therefore the optimal MAP decoder is still achievable by choosing $Y = [Y_{10} + Y_{00}, Y_{11} + Y_{01}]$ with $t_{10} = t_{00} = 1$ and $t_{11} = t_{01} = 2$.

(b) The mimimum energy signal set $\widetilde{\mathcal{W}} = \{\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ is obtained by subtracting the mean signal $m(t)$ from the w_i:

$$m = \frac{1}{4}\sum_i w_i = \frac{1}{2}\psi_0 + \frac{1}{2}\psi_1$$

$$\begin{aligned}
\tilde{w}_0 &= w_0 - m = \frac{1}{2}\psi_0 + \frac{1}{2}\psi_1, & \tilde{w}_1 &= w_1 - m = \frac{1}{2}\psi_0 - \frac{1}{2}\psi_1, \\
\tilde{w}_2 &= w_2 - m = -\frac{1}{2}\psi_0 + \frac{1}{2}\psi_1, & \tilde{w}_3 &= w_3 - m = -\frac{1}{2}\psi_0 - \frac{1}{2}\psi_1.
\end{aligned}$$



(c) Using basis $\mathcal{B}$ given above, the codewords associated to $\widetilde{\mathcal{W}}$ are

$$\tilde{c}_0 = \left[\frac{1}{2}, \frac{1}{2}\right], \quad \tilde{c}_1 = \left[\frac{1}{2}, -\frac{1}{2}\right], \quad \tilde{c}_2 = \left[-\frac{1}{2}, \frac{1}{2}\right], \quad \tilde{c}_3 = \left[-\frac{1}{2}, -\frac{1}{2}\right].$$

Recognizing a QAM constellation with minimum codeword distance $d = 1$, the error probability is given by

$$P_e(\widetilde{\mathcal{W}}) = 2Q\left(\frac{1}{\sqrt{2N_0}}\right) - Q\left(\frac{1}{\sqrt{2N_0}}\right)^2.$$

(d) $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are linked though an isometric transform, hence their error probabilities are identical:

$$P_e(\mathcal{W}) = P_e(\widetilde{\mathcal{W}}) = 2Q\left(\frac{1}{\sqrt{2N_0}}\right) - Q\left(\frac{1}{\sqrt{2N_0}}\right)^2.$$

PROBLEM 3. (a) By the Cauchy-Schwarz inequality:

$$\langle w_i, w_j\rangle^2 \leq \|w_i\|_2^2 \|w_j\|_2^2. \tag{6}$$

For $i \neq j$, eq. (6) simplifies to $\beta^2 \leq 1$, thus proving the claim.

(b) Based on the hint, we have

$$\begin{aligned}\|w_0 + w_1 + w_2\|_2^2 &= \|w_0\|_2^2 + \|w_1 + w_2\|_2^2 + 2\langle w_0, w_1 + w_2\rangle \\ &= \|w_0\|_2^2 + \|w_1\|_2^2 + \|w_2\|_2^2 + 2\langle w_0, w_1\rangle + 2\langle w_0, w_2\rangle + 2\langle w_1, w_2\rangle \\ &= 3 + 6\beta. \end{aligned} \tag{7}$$

By the non-negativity property of $\|\cdot\|_2$, we must have $3+6\beta \geq 0$, which implies $\beta \geq -\frac{1}{2}$.

(c) The mimimum energy signal set $\widetilde{\mathcal{W}} = \{\tilde{w}_0, \tilde{w}_1, \tilde{w}_2\}$ is obtained by subtracting the mean signal $m(t)$ from the w_i:

$$m = \frac{1}{3}\sum_i w_i.$$

$$\begin{aligned}\tilde{w}_0 &= w_0 - m = +\frac{2}{3}w_0 - \frac{1}{3}w_1 - \frac{1}{3}w_2, \\ \tilde{w}_1 &= w_1 - m = -\frac{1}{3}w_0 + \frac{2}{3}w_1 - \frac{1}{3}w_2, \\ \tilde{w}_2 &= w_2 - m = -\frac{1}{3}w_0 - \frac{1}{3}w_1 + \frac{2}{3}w_2.\end{aligned}$$

Let $\{w_i, w_j, w_k\}$ and $\{\tilde{w}_i, \tilde{w}_j, \tilde{w}_k\}$ be some arbitrary relabeling of $\{w_0, w_1, w_2\}$ and $\{\tilde{w}_0, \tilde{w}_1, \tilde{w}_2\}$ respectively:

$$\begin{aligned}\|\tilde{w}_i\|_2^2 &= \|w_i - m\|_2^2 = \|w_i\|_2^2 + \|m\|_2^2 - 2\langle w_i, m\rangle \\ &= \|w_i\|_2^2 + \left\|\frac{1}{3}w_i + \frac{1}{3}w_j + \frac{1}{3}w_k\right\|_2^2 - 2\langle w_i, \frac{1}{3}w_i + \frac{1}{3}w_j + \frac{1}{3}w_k\rangle \\ &\overset{eq.\ (7)}{=} \|w_i\|_2^2 + \frac{1}{9}(3+6\beta) - \frac{2}{3}\left(\|w_i\|_2^2 + \langle w_i, w_j\rangle + \langle w_i, w_k\rangle\right) \\ &= 1 + \frac{1}{9}(3+6\beta) - \frac{2}{3}(1+2\beta) = \frac{2}{3}(1-\beta) = E, \end{aligned} \tag{8}$$

$$\begin{aligned}\langle \tilde{w}_i, \tilde{w}_j\rangle &= \langle \frac{2}{3}w_i - \frac{1}{3}w_j - \frac{1}{3}w_k, -\frac{1}{3}w_i + \frac{2}{3}w_j - \frac{1}{3}w_k\rangle \\ &= -\frac{2}{9}\|w_i\|_2^2 + \frac{5}{9}\langle w_i, w_j\rangle - \frac{1}{9}\langle w_i, w_k\rangle - \frac{2}{9}\|w_j\|_2^2 - \frac{1}{9}\langle w_j, w_k\rangle + \frac{1}{9}\|w_k\|_2^2 \\ &= -\frac{1}{3}(1-\beta) \overset{eq.\ (8)}{=} -\frac{E}{2}. \end{aligned} \tag{9}$$

(d) $\widetilde{\mathcal{W}}$ is related to $\mathcal{W} = \{w_0, w_1, w_2\}$ through an isometric transform, hence they have the same error probability P_e. We will therefore consider $\widetilde{\mathcal{W}}$ below.

Notice from item c that $\|\tilde{w}_i\|_2^2 = E$ and $\frac{\langle \tilde{w}_i, \tilde{w}_j \rangle}{\|\tilde{w}_i\|_2 \|\tilde{w}_j\|_2} = -\frac{1}{2} = \cos(120°)$, therefore $\widetilde{\mathcal{W}}$ is indeed a 3-PSK constellation and we can quantify its error rate P_e using $e_3(\cdot)$. To this end, let us define an orthonormal basis $\mathcal{B} = \{\psi_0, \psi_1\}$ for $\widetilde{\mathcal{W}}$:

$$\psi_0 = \frac{\tilde{w}_0}{\|\tilde{w}_0\|_2} = \frac{\tilde{w}_0}{\sqrt{E}}, \qquad \psi_1 = \frac{\tilde{w}_1 - \langle \tilde{w}_1, \psi_0 \rangle \psi_0}{\|\tilde{w}_1 - \langle \tilde{w}_1, \psi_0 \rangle \psi_0\|_2} = \frac{\sqrt{E}}{2}\tilde{w}_0 + \tilde{w}_1.$$

Using $\mathcal{B}$, the sufficient statistic $Y \in \mathbb{R}^2$ behaves as

$$Y|H = 0 \sim \mathcal{N}\left(\left[\sqrt{E}, 0\right], \frac{N_0}{2} I_2\right).$$

Equivalently $\widetilde{Y} = \sqrt{\frac{2}{N_0}} Y$ behaves as

$$\widetilde{Y}|H = 0 \sim \mathcal{N}\left(\left[A, 0\right], I_2\right),$$

with $A = \sqrt{2E/N_0}$. Therefore communicating with $\mathcal{W}$ over an AWGN channel of power spectral density $\frac{N_0}{2}$ gives rise to an error rate

$$P_e = e_3\left(\sqrt{\frac{2E}{N_0}}\right) = e_3\left(\sqrt{\frac{4(1-\beta)}{3N_0}}\right).$$

Problem 4. (a) Conditioned on H and t_0, Y follows the Gaussian distribution

$$Y|H, t_0 \sim \mathcal{N}\left(\left(w_H * h\right)(t_0), \frac{N_0}{2}\|h\|_2^2\right), \tag{10}$$

$$\left(w_H * h\right)(t_0) = \int_{t_0 - \frac{1}{2}}^{t_0 + \frac{1}{2}} w_H(t)\, dt = \frac{(-1)^H}{2}. \tag{11}$$

Given the above, we have

$$P_e|H, t_0 = Q\left(\frac{\left|\left(w_H * h\right)(t_0)\right|}{\sqrt{N_0/2}\,\|h\|_2}\right) = Q\left(\sqrt{\frac{1}{2N_0}}\right),$$

$$P_e = \mathbb{E}_{H,t_0}\left[P_e|H, t_0\right] = Q\left(\sqrt{\frac{1}{2N_0}}\right).$$

(b) eq. (10) still holds here, but now $\|h\|_2 = \sqrt{2}$ and $\mathbb{E}\left[Y|H, t_0\right]$ is given by

$$\left(w_H * h\right)(t_0) = \int_{t_0 - 1}^{t_0 + 1} w_H(t)\, dt = (-1)^H.$$

The error probability therefore becomes

$$P_e|H, t_0 = Q\left(\frac{\left|\left(w_H * h\right)(t_0)\right|}{\sqrt{N_0/2}\,\|h\|_2}\right) = Q\left(\sqrt{\frac{1}{N_0}}\right),$$

$$P_e = \mathbb{E}_{H,t_0}\left[P_e|H, t_0\right] = Q\left(\sqrt{\frac{1}{N_0}}\right).$$

(c) eq. (10) still holds here, but we will simplify terms differently:

$$\|h\|_2^2 = \int_{-\infty}^{0} h^2(t)\,dt + \int_{0}^{\infty} h^2(t)\,dt = E_- + E_+,$$

$$\begin{aligned}(w_H * h)(0.5) &= \int w_H(t) h\left(\frac{1}{2} - t\right) dt = (-1)^H \int_{-\frac{1}{2}}^{\frac{1}{2}} h\left(\frac{1}{2} - t\right) dt \\ &= (-1)^H \int_0^1 h(\alpha)\,d\alpha = (-1)^H A_+,\end{aligned}$$

$$\begin{aligned}(w_H * h)(-0.5) &= \int w_H(t) h\left(-\frac{1}{2} - t\right) dt = (-1)^H \int_{-\frac{1}{2}}^{\frac{1}{2}} h\left(-\frac{1}{2} - t\right) dt \\ &= (-1)^H \int_{-1}^{0} h(\alpha)\,d\alpha = (-1)^H A_-.\end{aligned}$$

The error probability therefore becomes

$$P_e|H, t_0 = Q\left(\frac{|(w_H * h)(t_0)|}{\sqrt{N_0/2}\,\|h\|_2}\right) = \begin{cases} Q\left(\sqrt{\frac{2A_+^2}{N_0(E_-+E_+)}}\right), & t_0 = 0.5 \\ Q\left(\sqrt{\frac{2A_-^2}{N_0(E_-+E_+)}}\right), & t_0 = -0.5 \end{cases},$$

$$P_e = \mathbb{E}_{H,t_0}\left[P_e|H, t_0\right] = \frac{1}{2} Q\left(\sqrt{\frac{2A_+^2}{N_0\left(E_- + E_+\right)}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2A_-^2}{N_0\left(E_- + E_+\right)}}\right).$$

(d)

$$A_+^2 = \left(\int_0^1 h(t) \cdot 1\,dt\right)^2 \leq \int_0^1 h^2(t)\,dt \int_0^1 dt \leq \int_0^\infty h^2(t)\,dt = E_+,$$

where the first inequality is due to the Cauchy-Schwarz theorem. A similar argument applied to (A_-, E_-) also implies $A_-^2 \leq E_-$.

(e)

$$\begin{aligned}P_e &\overset{(c)}{=} \frac{1}{2} Q\left(\sqrt{\frac{A_+^2}{\sigma^2\left(E_- + E_+\right)}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{A_-^2}{\sigma^2\left(E_- + E_+\right)}}\right) \geq Q\left(\sqrt{\frac{A_-^2 + A_+^2}{2\sigma^2\left(E_- + E_+\right)}}\right) \\ &\overset{(d)}{\geq} Q\left(\sqrt{\frac{1}{2\sigma^2}}\right) = Q\left(\sqrt{\frac{1}{N_0}}\right).\end{aligned}$$

As the lower bound is achieved by the filter used in (b), the former is optimal.