

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 17
Homework 7

Information Theory and Coding
Nov. 9, 2021

PROBLEM 1. A source produces independent, equally probable symbols from an alphabet (a_1, a_2) at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a_1 as 000 and the source symbol a_2 as 111. If in the corresponding 3 second interval of the channel output, any of the sequences 000,001,010,100 is received, a_1 is decoded; otherwise, a_2 is decoded. Let $\epsilon < 1/2$ be the channel crossover probability.

- For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a_1 came out of the source given that received sequence.
- Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.
- Find the probability of an incorrect decision (using part (a) is not the easy way here).
- If the source is slowed down to produce one letter every $2n + 1$ seconds, a_1 being encoded by $2n + 1$ 0's and a_2 being encoded by $2n + 1$ 1's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \rightarrow \infty$.

PROBLEM 2. Consider two discrete memoryless channels. The first channel has input alphabet \mathcal{X} , output alphabet \mathcal{Y} ; the second channel has input alphabet \mathcal{Y} and output alphabet \mathcal{Z} . The first channel is described by the conditional probabilities $P_1(y|x)$ and the second channel by $P_2(z|y)$. Let the capacities of these channels be C_1 and C_2 . Consider a third memoryless channel described by probabilities

$$P_3(z|x) = \sum_{y \in \mathcal{Y}} P_2(z|y)P_1(y|x), \quad x \in \mathcal{X}, z \in \mathcal{Z}.$$

- Show that the capacity C_3 of this third channel satisfies

$$C_3 \leq \min\{C_1, C_2\}.$$

- A helpful statistician preprocesses the output of the first channel by forming $\tilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.
 - Show that he is wrong.
 - Under what conditions does he not strictly decrease the capacity?

PROBLEM 3. Consider a random source \mathcal{S} of information, and let W be a random variable which represents the first L symbols U_1, \dots, U_L of this source, i.e., $W = U_1^L$. We want to transmit the value of W using a memoryless stationary channel as follows:

- At time $t = 1$, we send $X_1 = f_1(W)$ through the channel.

- At time $t = i + 1 \leq n$, we send $X_{i+1} = f_i(W, Y^i)$ through the channel. Y_1, \dots, Y_i are the output of the channel at times $t = 1, \dots, i$ respectively,

f_1, \dots, f_n are n mappings that constitute the encoder. Clearly, this is a communication system with feedback as we are using the value of Y^i in the computation of X_{i+1} .

In the previous problem, we gave an example which satisfies $I(X^n; Y^n) > nC$ and $I(W; Y^n) \leq nC$. Show that the inequality $I(W; Y^n) \leq nC$ always holds by justifying each of the following equalities and inequalities:

$$\begin{aligned}
 I(W; Y^n) &\stackrel{(a)}{=} \sum_{i=1}^n I(W; Y_i | Y^{i-1}) \stackrel{(b)}{\leq} \sum_{i=1}^n I(W, Y^{i-1}; Y_i) \stackrel{(c)}{\leq} \sum_{i=1}^n I(W, X_i, X^{i-1}, Y^{i-1}; Y_i) \\
 &\stackrel{(d)}{=} \sum_{i=1}^n I(X_i, X^{i-1}, Y^{i-1}; Y_i) \stackrel{(e)}{=} \sum_{i=1}^n I(X_i; Y_i) \stackrel{(f)}{\leq} nC.
 \end{aligned}$$

Since $I(W; Y^n)$ represents the amount of information that is shared with the receiver, the inequality $I(W; Y^n) \leq nC$ shows that feedback does not increase the capacity.

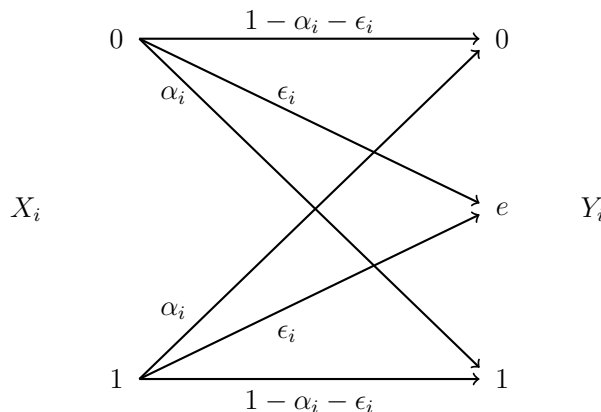
PROBLEM 4. *Channels with memory have higher capacity.* Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where \oplus is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$.

Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that Z_1, Z_2, \dots, Z_n are not necessarily independent. Assume that (Z_1, \dots, Z_n) is independent of the input (X_1, \dots, X_n) . Let $C = \log 2 - H(p, 1 - p)$. Show that

$$\max_{p_{X_1, X_2, \dots, X_n}} I(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n) \geq nC.$$

PROBLEM 5. Consider two discrete memoryless channels. The input alphabet, output alphabet, transition probabilities and capacity of the k 'th channel is given by $\mathcal{X}_k, \mathcal{Y}_k, p_k$ and C_k respectively ($k = 1, 2$). The channels operate independently. A communication system has access to both channels, that is, the effective channel between the transmitter and receiver has input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ and transition probabilities $p_1(y_1|x_1)p_2(y_2|x_2)$. Find the capacity of this channel.

PROBLEM 6. Consider the following symmetric channel with binary input that maps to a ternary output. (A channel that may either flip or erase the transmitted symbol.)



In other words,

$$p_i(y_i|0) = \begin{cases} 1 - \alpha_i - \epsilon_i, & y_i = 0 \\ \epsilon_i, & y_i = e \\ \alpha_i, & y_i = 1 \end{cases} \quad \alpha_i, \epsilon_i \in [0, 1], \quad \alpha_i + \epsilon_i \leq 1$$

and vice versa for $p_i(y_i|1)$. Also, Y_i 's are independent of each other given X_i 's. (i.e. $p(y_1^n|x_1^n) = \prod_{i=1}^n p_i(y_i|x_i)$ for any $n \geq 1$).

- (a) Suppose the channel is not time varying, that is $\alpha_i = \alpha$ and $\epsilon_i = \epsilon$. Find the capacity $C = \max_{p(x)} I(X; Y)$
- (b) What are the special cases when $\alpha = 0, \epsilon \neq 0$ and $\alpha \neq 0, \epsilon = 0$? What happens when $\alpha + \epsilon = 1$?
- (c) Now, suppose that the channel is time varying, that is, for each channel use α_i 's and ϵ_i 's differ. Find $\max_{p(x_1^n)} I(X_1^n; Y_1^n)$.