ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

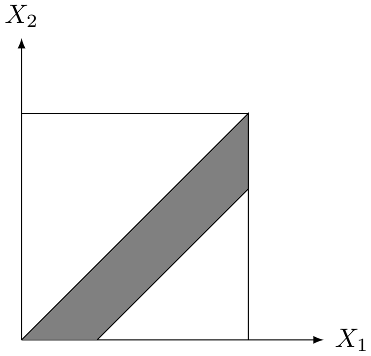
Handout 3 — Principles of Digital Communications

Solutions to Problem Set 1 — Feb. 23, 2024

SOLUTION 1. In each case, the shaded region represents the (X_1, X_2) values satisfying the corresponding inequalities. Since X_1 and X_2 are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use $\Pr\{\cdot\}$ to denote the probability of an event.

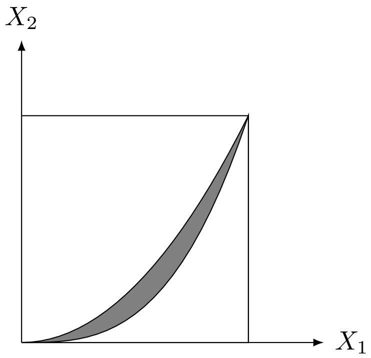
(a)

$$\Pr\left\{0 \le X_1 - X_2 \le \frac{1}{3}\right\} = \frac{1}{2} - \frac{1}{2} \times \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{5}{18}.$$



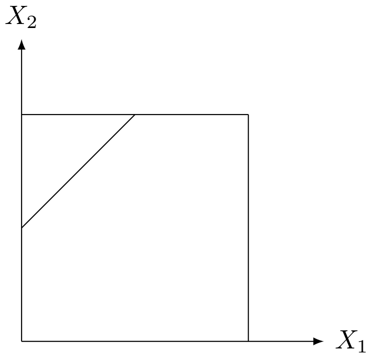
(b)

$$\Pr\left\{X_1^3 \le X_2 \le X_1^2\right\} = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



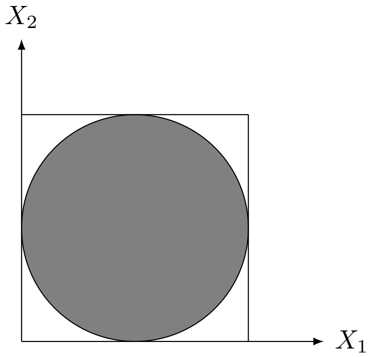
(c)

$$\Pr\left\{X_2 - X_1 = \frac{1}{2}\right\} = 0.$$



(d)

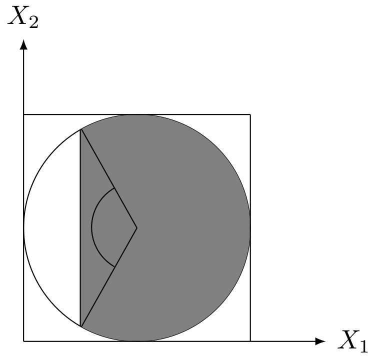
$$\Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\right\}=\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{4}.$$



(e) In this part we have

$$\begin{aligned}
&\Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2 \;\middle|\; X_1\ge\frac{1}{4}\right\}\\
&\quad=\frac{\Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\ ,\ X_1\ge\frac{1}{4}\right\}}{\Pr\left\{X_1\ge\frac{1}{4}\right\}}\\
&\quad=\frac{\frac{\pi}{6}+\frac{\sqrt{3}}{16}}{\frac{3}{4}}.
\end{aligned}$$

It can easily be seen that the probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area into two parts, triangular and sub-circular. It is easy to show that the drawn angle is 120°, so the sub-circular part consists of $\frac{2}{3}$ of the circle area. So the sub-circular part's area is $\frac{2}{3}\,\pi(\frac{1}{2})^2=\frac{\pi}{6}$ and the triangular part's area is $\frac{\sqrt{3}}{16}$. Summing the area of these two parts, we reach the final result.



SOLUTION 2.

(a) First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to

$$\Pr\{\text{All } k \text{ balls are black}\} = \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$\Pr\{\text{At least one ball is white}\} = 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

(b) Define the following random variables

$$X = \begin{cases} 0 & \text{if the chosen coin is fair,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$Y = \begin{cases} 00 & \text{if both outcomes are tail,} \\ 01 & \text{if the first one is tail, the second one is head,} \\ 10 & \text{if the first one is head, the second one is tail,} \\ 11 & \text{if both outcomes are head.} \end{cases}$$

Having defined these random variables, we want to compute $\Pr\{X = 0|Y = 11\}$. So we can write

$$\begin{aligned} \Pr\{X = 0|Y = 11\} &= \frac{\Pr\{Y = 11|X = 0\}\Pr\{X = 0\}}{\Pr\{Y = 11\}} \\ &= \frac{1/4 \times 1/2}{\Pr\{Y = 11\}} \\ &= \frac{1/8}{\Pr\{Y = 11\}}. \end{aligned}$$

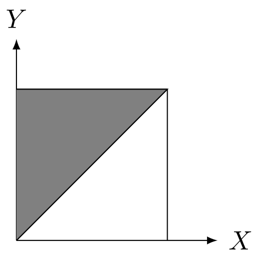
Then for $\Pr\{Y = 11\}$ we have

$$\begin{aligned} \Pr\{Y = 11\} &= \Pr\{X = 0\} \cdot \Pr\{Y = 11|X = 0\} + \Pr\{X = 1\} \cdot \Pr\{Y = 11|X = 1\} \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8. \end{aligned}$$

So, finally we have

$$\Pr\{X = 0|Y = 11\} = \frac{1/8}{5/8} = \frac{1}{5}.$$

SOLUTION 3. The probability mass has been distributed uniformly on the upper-triangular area according to the shape below:



(a) If X and Y were independent, then the distribution of X would not depend on Y. This is clearly not the case. In fact, the range of values taken by X is between 0 and Y.

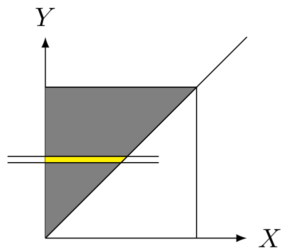
(b) The integral of $f_{X,Y}(x, y)$ must be 1. Hence $A \times \frac{1}{2} = 1$ and so $A = 2$.

(c) We know that $f_Y(y)\ dy = \Pr\{y < Y < y + dy\}$ but for a special y as can be seen from the figure below this probability mass is equal to A times the area of a rectangle with length y and width dy when $0 \le y \le 1$.

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Analytic evaluation gives

$$f_Y(y) = \int_0^1 f_{X,Y}(x, y)\ dx = \int_0^y 2\ dx = 2y.$$



(d) Under the condition $Y = y$, the random variable X is uniformly distributed between 0 and y and so $f(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.

(e) $f(Y)$ is a function of Y so it is a random variable and we can compute its expected value.

$$\mathbb{E}[f(Y)] = \int_0^1 f(y) f_Y(y)\ dy = \int_0^1 y^2\ dy = \frac{1}{3}.$$

(f) We compute $\mathbb{E}[X]$ using the definition.

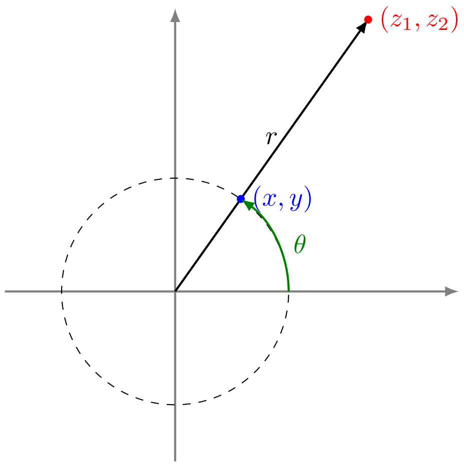
$$\mathbb{E}[X] = \iint x f_{X,Y}(x, y)\ dx\ dy = \int_0^1 \left[\int_0^y 2x\ dx \right]\ dy = \frac{1}{3},$$

and it is seen that $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$. This result, which holds in general, is named the law of total expectation.

Solution 4. It is easy to check $X^2 + Y^2 = 1$, hence (X, Y) is a point on the unit circle. As Z_1 and Z_2 are independent Gaussians, their joint probability density is

$$f_{Z_1,Z_2}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2) = \frac{1}{\sigma^2 2\pi} e^{-\frac{z_1^2+z_2^2}{2\sigma^2}}.$$

Note that this joint density depends on (z_1, z_2) only through its distance from the origin $r = \sqrt{z_1^2 + z_2^2}$, and not on the angle θ. Consequently, the angle Θ is uniformly distributed. Since (X, Y) is a scaled version of (Z_1, Z_2) without changing the angle, it is uniformly distributed on the circle (see the figure below).



A more formal justification of the fact that the angle is uniformly distributed is as follows. Let $R := \sqrt{Z_1^2 + Z_2^2}$ and $\Theta := \tan^{-1}(Z_2/Z_1)$ be the *polar* coordinates of the point (Z_1, Z_2). We can conclude that the joint density $f_{R,\Theta}(r, \theta)$ is only a function of r. Consequently,

$$f_\Theta(\theta) = \int_0^\infty f_{R,\Theta}(r, \theta)\, dr = C \qquad \forall\theta \in [0, 2\pi),$$

where C is a *constant* and (as $\int_0^{2\pi} f_\Theta(\theta)\, d\theta = 1$) is equal to $\frac{1}{2\pi}$. Thus, Θ has a uniform distribution in $[0, 2\pi)$.

Remark. *One can check that the random variable R has a Rayleigh distribution,*

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}.$$

Furthermore the random variables R and Θ are also independent.

Solution 5.

Recall that:

- By definition, X and Y are uncorrelated if and only if

$$0 = \operatorname{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

  Hence $\operatorname{cov}(X, Y) = 0$ is equivalent to the condition $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$.

- X and Y are independent when $f_{X,Y} = f_X f_Y$.

(a) Assume that the random variables X and Y are independent. Then:

$$\begin{aligned}\mathbb{E}[XY] &= \iint xy f_{X,Y}(x,y)\ dx\ dy = \iint xy f_X(x) f_Y(y)\ dx\ dy \\ &= \int x f_X(x)\ dx \int y f_Y(y)\ dy = \mathbb{E}[X]\,\mathbb{E}[Y],\end{aligned}$$

where the second equality follows from the assumption that X and Y are independent. Hence, if X and Y are independent, they are also uncorrelated.

(b) X and Y are obviously dependent. For example, $X = 0$ implies $U = 0$ and $V = 0$. Hence it implies also $Y = 0$. The marginals of X and Y are:

$$X = \begin{cases} 0, & \text{with prob. } \frac{1}{4} \\ 1, & \text{with prob. } \frac{1}{2} \\ 2, & \text{with prob. } \frac{1}{4}, \end{cases}$$

$$Y = \begin{cases} 0, & \text{with prob. } \frac{1}{2} \\ 1, & \text{with prob. } \frac{1}{2}. \end{cases}$$

The mean for X is $\mathbb{E}[X] = 1$ and for Y it is $\mathbb{E}[Y] = \frac{1}{2}$. Finally, we have that

$$\mathbb{E}[XY] = \frac{1}{4} \times 0 \times 0 + \frac{1}{4} \times 1 \times 1 + \frac{1}{4} \times 1 \times 1 + \frac{1}{4} \times 2 \times 0 = \frac{1}{2}.$$

From these two:

$$\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 0,$$

thus, X and Y are uncorrelated, even though they are dependent.

Solution 6. The unit sphere is defined by equation $x^2+y^2+z^2 = 1$. Hence $X^2+Y^2+Z^2 = 1$ with probability 1, which implies $\mathbb{E}[X^2 + Y^2 + Z^2] = 1$.

By linearity of expectation,

$$\mathbb{E}[X^2 + Y^2 + Z^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + \mathbb{E}[Z^2] = 1. \tag{*}$$

Furthermore, because of symmetry X, Y and Z have the same marginal distributions. In particular, $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = \mathbb{E}[Z^2]$. Using this in (*) we conclude that $\mathbb{E}[X^2] = \frac{1}{3}$.

Another possible solution is to change the coordinates such that $x = \sin(\Theta)$, $\Theta \in [-\pi/2, \pi/2]$, and $\ell = \cos(\Theta)$ is the radius of circle formed by the sphere surface when intersected with the plane of constant X. The CDF of these points is given by:

$$\begin{aligned}\mathbb{P}(\Theta \le \alpha) &= \frac{\text{Sphere Surface Area below } X = \sin(\alpha) \text{ Plane}}{\text{Sphere Surface Area}} \\ &= \frac{1}{4\pi} \int_{-\pi/2}^{\alpha} 2\pi \ell ds(\theta)\end{aligned}$$

where $ds(\theta)$ is the unit arc length at θ and given by $\sqrt{(dx)^2 + (d\ell)^2}$. A common mistake is to form a false analogy between the formula for the volume of revolution and the surface of revolution, thus forgetting the arc length of the surface area. Thus it gives us:

$$\mathbb{P}(\Theta \le \alpha) = \frac{1}{4\pi} \int_{-\pi/2}^{\alpha} 2\pi \cos(\theta) \underbrace{\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{d\ell}{d\theta}\right)^2}}_{=1} d\theta$$

Thus it gives us the PDF:

$$f_\Theta(\theta) = \frac{\cos(\theta)}{2}, \theta \in [-\pi/2, \pi/2]$$

Taking the expectation with this PDF:

$$\begin{aligned}
E[X^2] &= E[\sin^2(\theta)] \\
&= \int_{-\pi/2}^{\pi/2} \sin^2(\theta) f_\Theta(\theta) d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin^2(\theta) d\sin(\theta) \\
&= \frac{1}{2}\int_{-1}^{1} x^2 dx \\
&= \frac{1}{3}.
\end{aligned}$$

SOLUTION 7. (a) Given that $f_X(x) = e^{-x}$ and $f_{\hat{X}}(x) = 2e^{-2x}$ for $x \geq 0$, then $f_X(x) \leq f_{\hat{X}}(x)$ if and only if $x \leq \log(2)$.

(b) $\mathbb{P}\left(f_X(X) \leq f_{\hat{X}}(X)\right) = \mathbb{P}\left(X \leq \log(2)\right) = \int_0^{\log(2)} e^{-x} \mathrm{d}x = \frac{1}{2}$.

(c) $\mathbb{P}\left(f_X(\hat{X}) \geq f_{\hat{X}}(\hat{X})\right) = \mathbb{P}\left(\hat{X} \geq \log(2)\right) = \int_{\log(2)}^{\infty} 2e^{-2\hat{x}} \mathrm{d}\hat{x} = \frac{1}{4}$.