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Handout 9 Principles of Digital Communications
Solutions to Problem Set 4 Mar. 15, 2024

SOLUTION 1. If $H = 0$, we have $Y_2 = Z_1 Z_2 = Y_1 Z_2$, and if $H = 1$, we have $Y_2 = -Z_1 Z_2 = Y_1 Z_2$. Therefore, $Y_2 = Y_1 Z_2$ in all cases. Now since Z_2 is independent of H, we clearly have $H \to Y_1 \to (Y_1, Y_1 Z_2)$. Hence, Y_1 is a sufficient statistic.

SOLUTION 2.

(a) The MAP decoder $\hat{H}(y)$ is given by

$$\hat{H}(y) = \arg\max_i P_{Y|H}(y|i) = \begin{cases} 0 & \text{if } y = 0 \text{ or } y = 1 \\ 1 & \text{if } y = 2 \text{ or } y = 3. \end{cases}$$

$T(Y)$ takes two values with the conditional probabilities

$$P_{T|H}(t|0) = \begin{cases} 0.7 & \text{if } t = 0 \\ 0.3 & \text{if } t = 1 \end{cases} \qquad P_{T|H}(t|1) = \begin{cases} 0.3 & \text{if } t = 0 \\ 0.7 & \text{if } t = 1. \end{cases}$$

Therefore, the MAP decoder $\hat{H}(T(y))$ is

$$\hat{H}(T(y)) = \arg\max_i P_{T(Y)|H}(t|i) = \begin{cases} 0 & \text{if } t = 0 \quad (y = 0 \text{ or } y = 1) \\ 1 & \text{if } t = 1 \quad (y = 2 \text{ or } y = 3). \end{cases}$$

Hence, the two decoders are equivalent.

(b) We have

$$\Pr\{Y = 0|T(Y) = 0, H = 0\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 0\}}{\Pr\{T(Y) = 0|H = 0\}} = \frac{0.4}{0.7} = \frac{4}{7}$$

and

$$\Pr\{Y = 0|T(Y) = 0, H = 1\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 1\}}{\Pr\{T(Y) = 0|H = 1\}} = \frac{0.1}{0.3} = \frac{1}{3}.$$

Thus $\Pr\{Y = 0|T(Y) = 0, H = 0\} \neq \Pr\{Y = 0|T(Y) = 0, H = 1\}$, hence $H \to T(Y) \to Y$ is not true, although the MAP decoders are equivalent.

SOLUTION 3.

(a) The MAP decision rule can always be written as

$$\begin{aligned} \hat{H}(y) &= \arg\max_i f_{Y|H}(y|i) P_H(i) \\ &= \arg\max_i g_i(T(y)) h(y) P_H(i) \\ &= \arg\max_i g_i(T(y)) P_H(i). \end{aligned}$$

The last step is valid because $h(y)$ is a non-negative constant which is independent of i and thus does not give any further information for our decision.

(b) Let us define the event $\mathcal{B} = \{y : T(y) = t\}$. Then,

$$\begin{aligned} f_{Y|H,T(Y)}(y|i,t) &= \frac{f_{Y,T(Y)|H}(y,t|i)P_H(i)}{\int_{\mathcal{Y}} f_{Y,T(Y)|H}(y,t|i)P_H(i)dy} = \frac{f_{Y|H}(y|i)f_{T(Y)|Y,H}(t|y,i)}{\int_{\mathcal{Y}} f_{Y|H}(y|i)f_{T(Y)|Y,H}(t|y,i)dy} \\ &= \frac{f_{Y|H}(y|i)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} f_{Y|H}(y|i)dy}. \end{aligned}$$

If $f_{Y|H}(y|i) = g_i(T(y))h(y)$, then

$$\begin{aligned} f_{Y|H,T(Y)}(y|i,t) &= \frac{g_i(T(y))h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} g_i(T(y))h(y)dy} \\ &= \frac{g_i(t)h(y)\mathbb{1}_{\mathcal{B}}(y)}{g_i(t)\int_{\mathcal{B}} h(y)dy} \\ &= \frac{h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} h(y)dy}. \end{aligned}$$

Hence, we see that $f_{Y|H,T(Y)}(y|i,t)$ does not depend on i, so $H \to T(Y) \to Y$.

(c) Note that $P_{Y_k|H}(1|i) = p_i$, $P_{Y_k|H}(0|i) = 1 - p_i$ and

$$P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = P_{Y_1|H}(y_1|i)\cdots P_{Y_n|H}(y_n|i).$$

Thus, we have

$$P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = p_i^t(1-p_i)^{(n-t)},$$

where $t = \sum_k y_k$.

Choosing $g_i(t) = p_i^t(1-p_i)^{(n-t)}$ and $h(y) = 1$, we see that $P_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i)$ fulfills the condition in the question.

(d) Because $Y_1,\ldots,Y_n$ are independent,

$$\begin{aligned} f_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) &= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_k-m_i)^2}{2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\sum_{k=1}^{n}\frac{(y_k-m_i)^2}{2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\sum_{k=1}^{n} y_k^2}{2}}\ e^{nm_i(\frac{1}{n}\sum_{k=1}^{n} y_k\ -\ \frac{m_i}{2})}. \end{aligned}$$

Choosing $g_i(t) = e^{nm_i(t-\frac{m_i}{2})}$ and $h(y_1,\ldots,y_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\sum_{k=1}^{n} y_k^2}{2}}$, we see that

$$f_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|i) = g_i(T(y_1,\ldots,y_n))h(y_1,\ldots,y_n).$$

Hence the condition in the question is fulfilled.

Solution 4.

(a) Since the X_i are i.i.d, the joint probability density (or mass) function is

$$p(x_1, x_2, \ldots, x_n) = \left[\prod_{i=1}^{n} h(x_i)\right] \exp\left[c(\theta)\sum_{i=1}^{n} T(x_i) - nB(\theta)\right].$$

By the Fisher-Neyman factorization theorem, $\sum_{i=1}^{n} T(x_i)$ is a sufficient statistic, where $T(x_i)$ is a sufficient statistic for the random variable X_i.

(b) It's easier to work with single random variables and use the result from (a):

- $p_X(x) = \lambda \exp(-\lambda x) = \exp(-\lambda x - \log \frac{1}{\lambda})$
  $h(x) = 1$, $c(\theta) = -\theta$, $T(x) = x$, $B(\theta) = \log \frac{1}{\theta}$
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$
- $p_X(x) = \dfrac{1}{2\sigma} \exp\left(-\dfrac{|x-\mu|}{\sigma}\right) = \exp\left((-\frac{1}{\sigma})|x-\mu| - (\log(2\sigma))\right)$
  $h(x) = 1$, $c(\theta) = -\frac{1}{\theta}$, $T(x) = |x-\mu|$, $B(\theta) = \log(2\theta)$
  By (a), $\sum_{i=1}^{n} |x_i - \mu|$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$
- $p_X(x) = \dfrac{\lambda^x \exp(-\lambda)}{x!} = \dfrac{1}{x!} \exp\left(\log(\lambda)x - \lambda\right)$
  $h(x) = \frac{1}{x!}$, $c(\theta) = \log\theta$, $T(x) = x$, $B(\theta) = \theta$
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$
- $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \exp\left(\log\left(\frac{p}{1-p}\right)x - n\log\frac{1}{1-p}\right)$
  $h(x) = \binom{n}{x}$, $c(\theta) = \log\frac{p}{1-p}$, $B(\theta) = n\log\frac{1}{1-p}$
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$

Solution 5.

(a) Inequality (a) follows from the *Bhattacharyya Bound.*

Using the definition of DMC, it is straightforward to see that

$$\begin{aligned} P_{Y|X}(y|c_0) &= \prod_{i=1}^{n} P_{Y|X}(y_i|c_{0,i}) \qquad \text{and} \\ P_{Y|X}(y|c_1) &= \prod_{i=1}^{n} P_{Y|X}(y_i|c_{1,i}). \end{aligned}$$

(b) follows by substituting the above values in (a).

Equality (c) is obtained by observing that $\sum_y$ is the same as $\sum_{y_1,\ldots,y_n}$ (the first one being a vector notation for the sum over all possible $y_1, \ldots, y_n$).

In (c), we see that we want the sum of all possible products. This is the same as summing over each y_i and taking the product of the resulting sum for all y_i. This results in equality (d). We obtain (e) by writing (d) in a more concise form.

When $c_{0,i} = c_{1,i}$, $\sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = P_{Y|X}(y|c_{0,i})$. Therefore,

$$\sum_y \sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = \sum_y P_{Y|X}(y|c_{0,i}) = 1.$$

This does not affect the product, so we are only interested in the terms where $c_{0,i} \neq c_{1,i}$. We form the product of all such sums where $c_{0,i} \neq c_{1,i}$. We then look out for terms where $c_{0,i} = a$ and $c_{1,i} = b, a \neq b$, and raise the sum to the appropriate power. (Eg. If we have the product $prpqrpqrr$, we would write it as $p^3q^2r^4$). Hence equality (f).

(b) For a binary input channel, we have only two source symbols $\mathcal{X} = \{a, b\}$. Thus,

$$\begin{aligned} P_e &\leq z^{n(a,b)} z^{n(b,a)} \\ &= z^{n(a,b)+n(b,a)} \\ &= z^{d_H(c_0,c_1)}. \end{aligned}$$

(c) The value of z is:

(i) For a binary input Gaussian channel,

$$\begin{aligned} z &= \int_y \sqrt{f_{Y|X}(y|0) f_{Y|X}(y|1)}\, dy \\ &= \exp\left(-\frac{E}{2\sigma^2}\right). \end{aligned}$$

(ii) For the Binary Symmetric Channel (BSC),

$$\begin{aligned} z &= \sqrt{\Pr\{y=-1|x=-1\}\Pr\{y=-1|x=1\}} + \sqrt{\Pr\{y=1|x=-1\}\Pr\{y=1|x=1\}} \\ &= 2\sqrt{\delta(1-\delta)}. \end{aligned}$$

(iii) For the Binary Erasure Channel (BEC),

$$\begin{aligned} z &= \sqrt{\Pr\{y=-1|x=-1\}\Pr\{y=-1|x=1\}} + \sqrt{\Pr\{y=E|x=-1\}\Pr\{y=E|x=1\}} \\ &\quad + \sqrt{\Pr\{y=1|x=-1\}\Pr\{y=1|x=1\}} \\ &= 0 + \delta + 0 \\ &= \delta. \end{aligned}$$

Solution 6.

$$\begin{aligned} P_{00} &= \Pr\{(N_1 \geq -a) \cap (N_2 \geq -a)\} \\ &= \Pr\{(N_1 \leq a)\} \Pr\{(N_2 \leq a)\} \\ &= \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2. \end{aligned}$$

By symmetry:

$$\begin{aligned} P_{01} = P_{03} &= \Pr\{(N_1 \leq -(2b-a)) \cap (N_2 \geq -a)\} \\ &= \Pr\{N_1 \geq 2b-a\} \Pr\{N_2 \leq a\} \\ &= Q\left(\frac{2b-a}{\sigma}\right)\left[1 - Q\left(\frac{a}{\sigma}\right)\right]. \end{aligned}$$

$$\begin{aligned} P_{02} &= \Pr\{(N_1 \leq -(2b-a)) \cap (N_2 \leq -(2b-a))\} \\ &= \Pr\{N_1 \geq 2b-a\} \Pr\{N_2 \geq 2b-a\} \\ &= \left[Q\left(\frac{2b-a}{\sigma}\right)\right]^2. \end{aligned}$$

$$
\begin{aligned}
P_{0\delta} &= 1 - \Pr\{(Y \in \mathcal{R}_0) \cup (Y \in \mathcal{R}_1) \cup (Y \in \mathcal{R}_2) \cup (Y \in \mathcal{R}_3) | c_0 \text{ was sent}\} \\
&= 1 - P_{00} - P_{01} - P_{02} - P_{03} \\
&= 1 - \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2 - 2Q\left(\frac{2b-a}{\sigma}\right)\left[1 - Q\left(\frac{a}{\sigma}\right)\right] - \left[Q\left(\frac{2b-a}{\sigma}\right)\right]^2 \\
&= 1 - \left[1 - Q\left(\frac{a}{\sigma}\right) + Q\left(\frac{2b-a}{\sigma}\right)\right]^2 .
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
P_{0\delta} &= \Pr\{(N_1 \in [a, 2b-a]) \cup (N_2 \in [a, 2b-a])\} \\
&= \Pr\{N_1 \in [a, 2b-a]\} + \Pr\{N_2 \in [a, 2b-a]\} - \Pr\{(N_1 \in [a, 2b-a]) \cap (N_2 \in [a, 2b-a])\} \\
&= 2\left[Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right] - \left[Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right]^2 ,
\end{aligned}
$$

which gives the same result as before.