ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 10 | Principles of Digital Communications

Problem Set 5 | Mar. 20, 2024

PROBLEM 1. Let R and Φ be independent random variables. R is distributed uniformly over the unit interval, Φ is distributed uniformly over the interval $[0, 2\pi)$.

(a) Interpret R and Φ as the polar coordinates of a point in the plane. It is clear that the point lies inside (or on) the unit circle. Is the distribution of the point uniform over the unit disk? Take a guess!

(b) Define the random variables

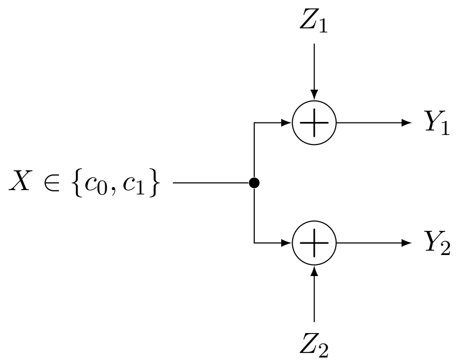
$$X = R\cos\Phi$$
$$Y = R\sin\Phi$$

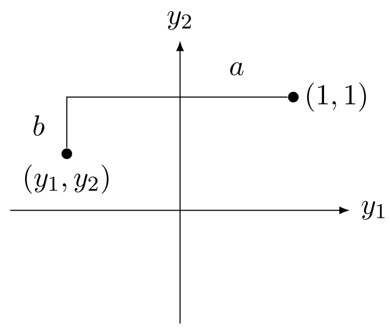
Find the joint distribution of the random variables X and Y by using the Jacobian determinant.

(c) Does the result of part (b) support or contradict your guess from part (a)? Explain.

PROBLEM 2. One of the two signals $c_0 = -1$, $c_1 = 1$ is transmitted over the channel shown in the left figure below. The two noise random variables Z_1 and Z_2 are statistically independent of the transmitted signal and of each other. Their density functions are

$$f_{Z_1}(\alpha) = f_{Z_2}(\alpha) = \frac{1}{2}e^{-|\alpha|}$$





(a) Derive a maximum likelihood decision rule.

(b) Describe the maximum likelihood decision regions in the (y_1, y_2) plane. Describe also the "either choice" regions, i.e., the regions where it does not matter if you decide for c_0 or for c_1.

Hint: Use geometric reasoning and the fact that for a point (y_1, y_2) as shown in the right figure above, $|y_1 - 1| + |y_2 - 1| = a + b$.

(c) A receiver decides that c_1 was transmitted if and only if $(y_1 + y_2) > 0$. Does this receiver minimize the error probability for equally likely messages?

(d) What is the error probability of the receiver in (c)?
Hint: One way to do this is to use the fact that if $W = Z_1 + Z_2$, then $f_W(w) = \frac{e^{-\omega}}{4}(1+\omega)$ for $\omega > 0$ and $f_W(-\omega) = f_W(\omega)$.

PROBLEM 3.

Let $\mathcal{W} = \{w_0(t), w_1(t)\}$ be the signal constellation used to communicate an equiprobable bit across an additive Gaussian noise channel. In this exercise, we verify that the projection of the channel output onto the inner product space $\mathcal{V}$ spanned by $\mathcal{W}$ is not necessarily a sufficient statistic, unless the noise is white.
Let $\psi_1(t), \psi_2(t)$ be an orthonormal basis for $\mathcal{V}$. We choose the additive noise to be $N(t) = Z_1\psi_1(t) + Z_2\psi_2(t) + Z_3\psi_3(t)$ for some normalized $\psi_3(t)$ that is orthogonal to $\psi_1(t)$ and $\psi_2(t)$, and choose Z_1, Z_2, Z_3 to be zero-mean jointly Gaussian random variables of identical variance σ^2. Let $c_i = (c_{i,1}, c_{i,2}, 0)^\mathsf{T}$ be the codeword associated to $w_i(t)$ with respect to the extended orthonormal basis $\psi_1(t), \psi_2(t), \psi_3(t)$. There is a one-to-one correspondence between the channel output $R(t)$ and $Y = (Y_1, Y_2, Y_3)^\mathsf{T}$, where $Y_i = \langle R, \psi_i \rangle$. In terms of Y, the hypothesis testing problem is

$$H = i : Y = c_i + Z, \quad i = \{0, 1\},$$

where we have defined $Z = (Z_1, Z_2, Z_3)^\mathsf{T}$.

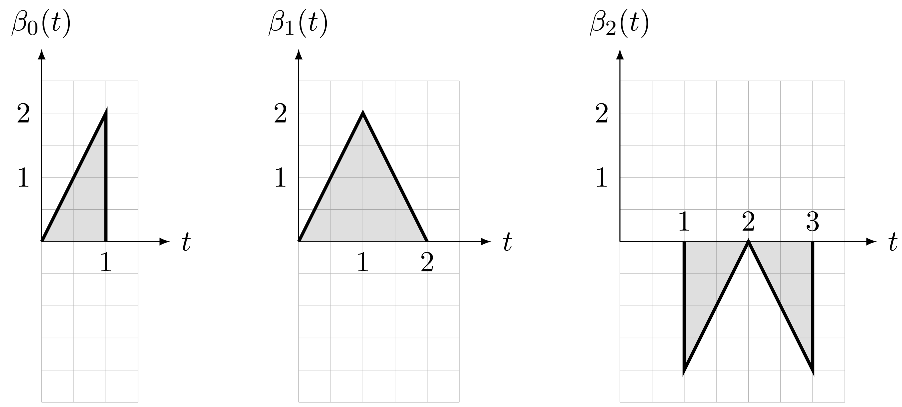
(a) As a warm-up exercise, let us first assume that Z_1, Z_2, Z_3 are independent. Use the Fisher–Neyman factorization theorem to show that $(Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) Now assume that Z_1 and Z_2 are independent, but $Z_3 = Z_2$. Prove that in this case $(Y_1, Y_2)^\mathsf{T}$ is *not* a sufficient statistic.

(c) To check a specific case, consider $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$. Determine the error probability of an ML receiver that observes $(Y_1, Y_2)^\mathsf{T}$ and that of another ML receiver that observes $(Y_1, Y_2, Y_3)^\mathsf{T}$.

PROBLEM 4.

(a) By means of the Gram–Schmidt procedure, find an orthonormal basis for the space spanned by the waveforms $\{\beta_0(t), \beta_1(t), \beta_2(t)\}$ below.

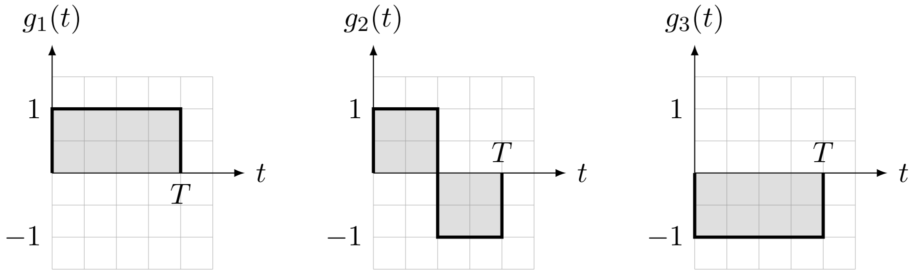


(b) In your chosen orthonormal basis, let $w_0(t)$ and $w_1(t)$ be represented by the codewords $c_0 = (3, -1, 1)^\mathsf{T}$ and $c_1 = (-1, 2, 3)^\mathsf{T}$ respectively. Plot $w_0(t)$ and $w_1(t)$.

(c) Compute the (standard) inner products $\langle c_0, c_1 \rangle$ and $\langle w_0, w_1 \rangle$ and compare them.

(d) Compute the norms $\|c_0\|$ and $\|w_0\|$ and compare them.

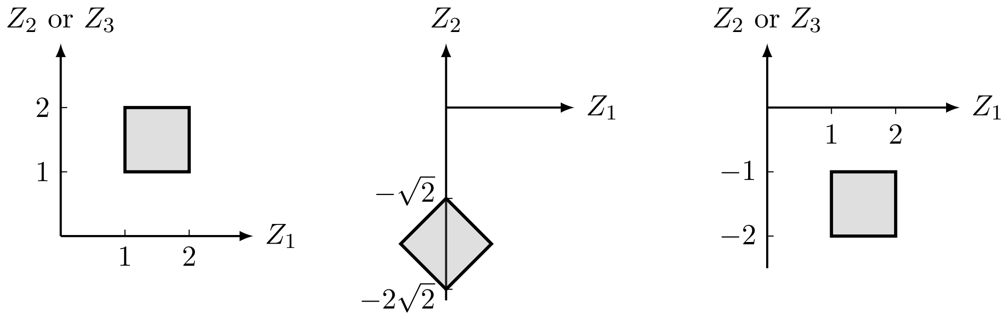
PROBLEM 5. Let $N(t)$ be white Gaussian noise of power spectral density $\frac{N_0}{2}$. Let $g_1(t)$, $g_2(t)$, and $g_3(t)$ be waveforms as shown below. For $i = 1, 2, 3$, let $Z_i = \int N(t) g_i^*(t) dt$, $Z = (Z_1, Z_2)^{\mathsf{T}}$, and $U = (Z_1, Z_3)^{\mathsf{T}}$.



(a) Determine the norm $\|g_i\|$, $i = 1, 2, 3$.

(b) Are Z_1 and Z_2 independent? Justify your answer.

Consider now the regions depicted below:



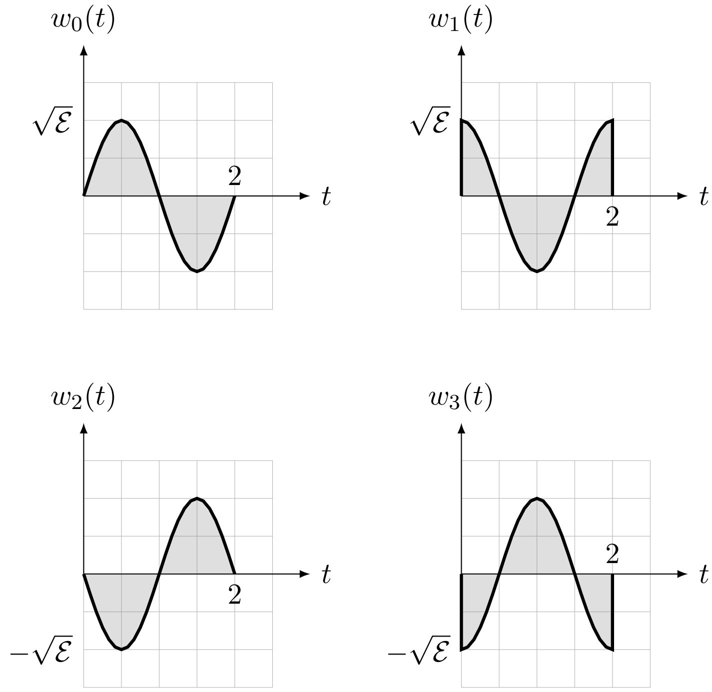
(c) Find the probability P_a that Z lies in the square of the left figure.

(d) Find the probability P_b that Z lies in the square of the middle figure.

(e) Find the probability Q_a that U lies in the square of the left figure.

(f) Find the probability Q_b that U lies in the square of the right figure.

Problem 6. Consider the four sinusoid waveforms $w_k(t)$, $k = 0, 1, 2, 3$ represented in the figure below.



(a) Determine an orthonormal basis for the signal space spanned by these waveforms.
Hint: No lengthy calculations needed.

(b) Determine the codewords c_i, $i = 0, 1, 2, 3$ representing the waveforms.

(c) Assume a transmitter sends w_i to communicate a digit $i \in \{0, 1, 2, 3\}$ across a continuous-time AWGN channel of power spectral density $\frac{N_0}{2}$. Write an expression for the error probability of the ML receiver in terms of $\mathcal{E}$ and N_0.