Handout 13 Principles of Digital Communications
Solutions to Problem Set 6 Mar. 29, 2024

SOLUTION 1.

(a) We have a binary hypothesis testing problem: The hypothesis H is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let H take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$\Pr\{H = 1|\hat{H}_L = 1, \hat{H}_R = 2\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \Pr\{H = 2|\hat{H}_L = 1, \hat{H}_R = 2\}$$

From the problem setting we know the priors $\Pr\{H = 1\}$ and $\Pr\{H = 2\}$; we can also determine the conditional probabilities $\Pr\{\hat{H}_L = 1|H = 1\}$, $\Pr\{\hat{H}_L = 1|H = 2\}$, $\Pr\{\hat{H}_R = 2|H = 1\}$ and $\Pr\{\hat{H}_R = 2|H = 2\}$ (we have $\Pr\{\hat{H}_L = 1|H = 1\} = 0.9$ and $\Pr\{\hat{H}_L = 1|H = 2\} = 0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2|H = 1\}\Pr\{H = 1\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \frac{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2|H = 2\}\Pr\{H = 2\}}{\Pr\{\hat{H}_L = 1, \hat{H}_R = 2\}}$$

Now, assuming that the event $\{\hat{H}_L = 1\}$ is independent of the event $\{\hat{H}_R = 2\}$ and simplifying the expression, we obtain

$$\begin{aligned}\Pr\{\hat{H}_L = 1|H = 1\}\Pr\{\hat{H}_R = 2|H = 1\}\Pr\{H = 1\} &\underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \\ \Pr\{\hat{H}_L = 1|H = 2\}\Pr\{\hat{H}_R = 2|H = 2\}\Pr\{H = 2\},&\end{aligned}$$

which is our final decision rule.

(b) Evaluating the previous decision rule, we have

$$0.9 \times 0.3 \times 0.25 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.0525$$

This implies that the answer $\hat{H}$ is equal to 1.

SOLUTION 2.

(a) We can write the MAP decision rule in the following way:

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

Plugging in, we find

$$\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \frac{p_0}{1-p_0},$$

and then

$$\left(\frac{\lambda_1}{\lambda_0}\right)^y \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}$$

Taking logarithms on both sides does not change the direction of the inequalities, therefore

$$y \log\left(\frac{\lambda_1}{\lambda_0}\right) \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)$$

Attention: the term $\log(\lambda_1/\lambda_0)$ can be negative, and if it is, then dividing by it involves changing the direction of the inequality.

Suppose $\lambda_1 > \lambda_0$. Then, $\log(\lambda_1/\lambda_0) > 0$, and the decision rule becomes

$$y \overset{\hat{H}=1}{\underset{\hat{H}=0}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \overset{\text{def}}{=} \theta$$

(b) We compute

$$\begin{aligned} P_e(0) &= \Pr\{Y > \theta | H = 0\} = \sum_{y=\lceil\theta\rceil}^{\infty} P_{Y|H}(y|0) \\ &= 1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}, \end{aligned}$$

and by analogy

$$\begin{aligned} P_e(1) &= \Pr\{Y < \theta | H = 1\} = \sum_{y=0}^{\lfloor\theta\rfloor} P_{Y|H}(y|1) \\ &= \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \end{aligned}$$

Thus, the probability of error becomes

$$P_e = p_0 \left(1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}\right) + (1-p_0) \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}$$

Now, suppose that $\lambda_1 < \lambda_0$. Then, $\log(\lambda_1/\lambda_0) < 0$, and we have to swap the inequality sign, thus

$$y \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1 - \lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

The rest of the analysis goes along the same lines, and finally, we obtain

$$P_e = p_0 \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1-p_0)\left(1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}\right)$$

The case $\lambda_0 = \lambda_1$ yields $\log(\lambda_1/\lambda_0) = 0$, so the decision rule becomes $0 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \theta$, regardless of y. Thus, we can exclude the case $\lambda_0 = \lambda_1$ from our discussion.

(c) Here, we are in the case $\lambda_1 > \lambda_0$, and we find $\theta \approx 4.54$. We thus evaluate

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{4} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{4} \left(\frac{10^y}{y!} e^{-10}\right) \approx 0.03705$$

(d) We find $\theta \approx 7.5163$

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{7} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{7} \left(\frac{20^y}{y!} e^{-20}\right) \approx 0.000885$$

The two Poisson distributions are much better separated than in (c); therefore, it becomes considerably easier to distinguish them based on one single observation y.

Solution 3. We use the Fisher–Neyman factorization theorem.

(a) Since Y is an i.i.d. sequence,

$$\begin{aligned} P_{Y|H}(y|i) = \prod_{k=1}^{n} P_{Y_k|H}(y_k|i) &= \frac{\lambda_i^{\sum_{k=1}^n y_k}}{\prod_{k=1}^n (y_k)!} e^{-n\lambda_i} \\ &= \underbrace{e^{-n\lambda_i} \lambda_i^{n\left(\frac{1}{n}\sum_{k=1}^n y_k\right)}}_{g_i(T(y))} \underbrace{\frac{1}{\prod_{k=1}^n (y_k)!}}_{h(y)} \end{aligned}$$

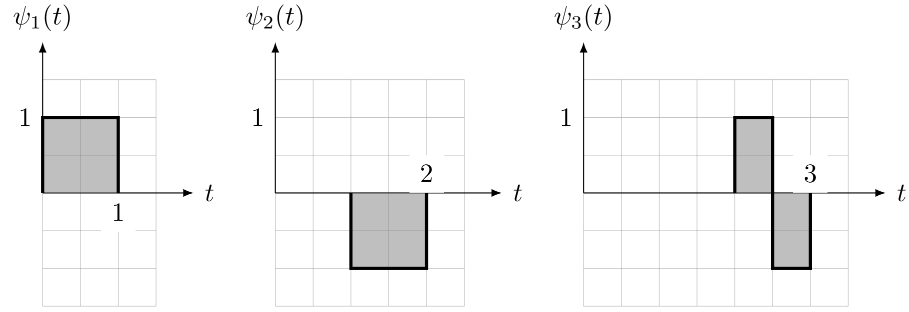
(b) Since $Z_1, \ldots, Z_n$ are i.i.d. additive noise samples,

$$\begin{aligned} f_{Y|H}(y|i) = \prod_{k=1}^{n} f_{Z_k|H}(y_k - \theta_i) &= \prod_{k=1}^{n} \lambda_i e^{-\lambda_i(y_k - \theta_i)} \mathbb{1}\{y_k \geq \theta_i\} \\ &= \underbrace{\lambda_i^n e^{n\lambda_i\theta_i} e^{-n\lambda_i\left(\frac{1}{n}\sum_{k=1}^n y_k\right)} \mathbb{1}\left\{\min\{y_1, \ldots, y_n\} \geq \theta_i\right\}}_{g_i(T(y))} \end{aligned}$$

with $h(y) = 1$.

Solution 4.

(a) It is straightforward to check that $w_0(t)$ has unit norm, i.e., $\|w_0(t)\| = 1$, thus $\psi_1(t) = w_0(t)$. With $\psi_1(t)$ we can reproduce the first portion of $w_1(t)$ (for t between 0 and 1). With $\psi_2(t)$ we need to be able to describe the remaining part of $w_1(t)$. Clearly $\psi_2(t)$ is as illustrated below. With $\psi_1(t)$ and $\psi_2(t)$ we also describe the part of $w_2(t)$ between $t = 0$ and $t = 2$. Hence $\psi_3(t)$ is selected as the unit-norm function that matches the part of $w_2(t)$ between $t = 2$ and $t = 3$. We immediately see that $w_3(t)$ is also a linear combination of $\psi_i(t)$, $i = 1, 2, 3$.



(b) Using the basis $\{\psi_1(t), \psi_2(t), \psi_3(t)\}$, one can give the following representation for the waveforms $w_i(t)$, $i = 0, \ldots, 3$:

$$w_0 = (1, 0, 0)^\mathsf{T}, w_1 = (-1, 1, 0)^\mathsf{T}, w_2 = (1, 1, 1)^\mathsf{T}, w_3 = (1, 1, -1)^\mathsf{T}$$

Solution 5.

(a) The optimal solution is to pass $R(t)$ through the matched filter $w(T - t)$ and sample the result at $t = T$ to get a sufficient statistic denoted by Y. (In this problem, $T = 1$.) Note that $Y = S + N$, where S and N are random variables denoting the signal and the noise components respectively. Under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $3c$, c, $-c$ and $-3c$ respectively.

Let $\hat{X}$ be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of $\hat{X}$ in the following fashion:

$$\hat{X} = \begin{cases} +3, & Y \in [2c, \infty) \\ +1, & Y \in [0, 2c) \\ -1, & Y \in [-2c, 0) \\ -3, & Y \in [-\infty, -2c) \end{cases} \tag{1}$$

(b) The probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4}\left[Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + Q\left(\frac{c}{\sqrt{N_0/2}}\right)\right] \\ &= \frac{3}{2} Q\left(\frac{c}{\sqrt{N_0/2}}\right) \end{aligned}$$

(c) In this case under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \dots, \alpha_3$ are $\frac{9c}{4}$, $\frac{3c}{4}$, $\frac{-3c}{4}$ and $\frac{-9c}{4}$ respectively. Using the decision rule in (1), the probability of error is given by

$$\begin{aligned}
P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\
&= \frac{1}{4}\left[Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right)\right. \\
&\quad \left. + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{c/4}{\sqrt{N_0/2}}\right)\right] \\
&= \frac{1}{2}\left[Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right)\right]
\end{aligned}$$

(d) The noise process $N(t)$ is a stationary Gaussian random process. So the noise component N (which is the sample of match-filter output at time T) is a Gaussian random variable with mean

$$\mathbb{E}[N] = \mathbb{E}\left[\int_{-\infty}^{\infty} N(t)w(t)dt\right] = \mathbb{E}\left[\int_{0}^{1} N(t)dt\right] = 0$$

Because the process $N(t)$ is stationary, without loss of generality we choose the boundaries of the integral to be 0 and T where in this problem $T = 1$.

Now, let us calculate the noise variance.

$$\begin{aligned}
\text{var}(N) &= \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \mathbb{E}[N^2] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} N(t)w(t)dt \int_{-\infty}^{\infty} N(v)w(v)dv\right] \\
&= \mathbb{E}\left[\int_{0}^{1} N(t)dt \int_{0}^{1} N(v)dv\right] \\
&= \mathbb{E}\left[\int_{0}^{1}\int_{0}^{1} N(t)N(v)dtdv\right] \\
&= \int_{0}^{1}\int_{0}^{1} K_N(t-v)dtdv \\
&= \int_{0}^{1}\int_{0}^{1} \frac{1}{4\alpha} e^{-|t-v|/\alpha} dtdv \\
&= \frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)
\end{aligned}$$

Thus the new probability of error is given by

$$\begin{aligned}
P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\
&= \frac{1}{4}\left[Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right) + Q\left(\frac{c}{\sqrt{\text{var}(N)}}\right)\right] \\
&= \frac{3}{2}Q\left(\frac{c}{\sqrt{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}}\right)
\end{aligned}$$