ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 31 Principles of Digital Communications
Final exam June 24, 2024

4 problems, 42 points, 180 minutes.
2 sheets (4 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (8 points)

Suppose $c_1, \dots, c_m$ (all in $\mathbb{R}^n$) are the codewords of a communication system with two receivers. Receiver A observes (A_1, A_2) where $A_1 = c_i + Z_1$, $A_2 = Z_1 + Z_2$, Receiver B observes $B_1 = c_i + Z_1$, $B_2 = c_i - Z_2$, where $Z_1 \in \mathbb{R}^n$ and $Z_2 \in \mathbb{R}^n$ are additive noises.

(a) (2 pts) Show that receiver A and receiver B have the same error probability (assuming both implement the optimal decision rule).
Hint: Show that receiver A can form the observation of receiver B and vice versa.

For the rest of the problem suppose Z_1 and Z_2 are independent and both are $\mathcal{N}(0, \sigma^2 I_n)$.

(b) (2 pts) Show that $U = (Z_1 - Z_2)/2$ and $W = (Z_1 + Z_2)/2$ are independent.

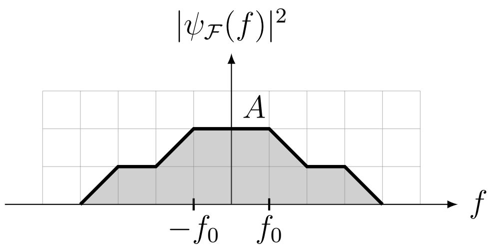
(c) (2 pts) Show that for receiver B, $T = (B_1 + B_2)/2$ is a sufficient statistic.

Consider now a receiver C that observes B_1 but with noise Z_1 being $\mathcal{N}(0, \tau^2 I_n)$.

(d) (2 pts) Can you find a τ_0 such that receiver C will perform better/worse than receiver B when τ^2 is less/more than τ_0^2?
Hint: Clearly τ_0 should depend on σ.

Problem 2. (10 points)

Suppose a pulse $\psi(t)$ has Fourier transform $\psi_{\mathcal{F}}(f)$, with $|\psi_{\mathcal{F}}(f)|^2$ sketched as below.



(a) (3 pts) Find the smallest positive T and the corresponding $A = |\psi_{\mathcal{F}}(0)|^2$ that will make $\big(\psi_j(t) = \psi(t - jT) : j \in \mathbb{Z}\big)$ an orthonormal collection of waveforms.

With T and A as in (a), let $w(t) = \sum_{j\in\mathbb{Z}} \sqrt{\mathcal{E}_s} X_j \psi(t - jT)$ and $W(t) = w(t + \Theta)$ where Θ is uniform in $[0, T]$ and independent of $\big(X_j : j \in \mathbb{Z}\big)$.

(b) (3 pts) Suppose X_j are i.i.d., with $\Pr(X_j = 1) = \Pr(X_j = -1) = 1/2$. Sketch the power spectral density $S_W(f)$ of $W(t)$. Which (if any) of the values $S_W(0)$, $S_W\big(1/(2T)\big)$, $S_W\big(1/T\big)$ are equal to 0?

Suppose we are requested to ensure $S_W(0) = 0$. A colleague suggests setting $X_j = B_j - B_{j-1}$ where B_j are i.i.d., with $\Pr(B_j = 0) = \Pr(B_j = 1) = 1/2$.

(c) (2 pts) With $\big(X_j : j \in \mathbb{Z}\big)$ as above, find $K_X[k] = \mathbb{E}[X_j X_{j+k}]$.

(d) (2 pts) Find $S_W(f)$ with this $\big(X_j : j \in \mathbb{Z}\big)$. Does the suggestion of our colleague work?

Problem 3. (13 points)
Suppose the bit stream $b_1, b_2, \ldots$ with $b_i = \pm 1$ is encoded by a convolutional encoder to the symbol stream $x_1, x_2, \ldots$ via

$$x_{2j-1} = b_j b_{j-2}, \quad x_{2j} = b_j b_{j-1} b_{j-2}$$

(as in the running-example used in the book and class). In computing the x's we assume $b_0 = b_{-1} = +1$. Recall that $T(I, D)$ for this encoder is $ID^5/(1 - 2ID)$.
Suppose that the receiver receives a sequence y_i where $y_i = x_i$ with probability $1 - p$ and $y_i = 0$ with probability p. Which of these two alternatives happens is chosen independently for each i. Observe that if $y_i \neq 0$, then the receiver is sure that $x_i = y_i$.

(a) (3 pts) Draw a Trellis section that describes the encoder map.

(b) (2 pts) Describe the Viterbi decoder, by providing the following information: given the received sequence $y_1, y_2, \ldots$, we associate the branch labeled (x_{2j-1}, x_{2j}) in the trellis a metric given by _____, and the corresponding path metric is to be *maximized*.

(c) (3 pts) Suppose $y_1, y_2, y_3, y_4, y_5, y_6 = 0, 0, -1, -1, 0, +1$. What is the Viterbi-decoded sequence $\hat{b}_1, \hat{b}_2, \hat{b}_3$?

(d) (2 pts) Suppose the true bit sequence $(b_1, \ldots, b_k)$ is the all-plus sequence. Observe that this is encoded as an all-plus x sequence $(x_1, \ldots, x_n)$, and thus the received sequence contains only 0's and +1's. Suppose $b' = (b'_1, \ldots, b'_k)$ an incorrect bit sequence, with encoding $(x'_1, \ldots, x'_n)$. Show that the probability that the Viterbi decoder will decide b' is at most p^d where d is the number of -1's in $(x'_1, \ldots, x'_n)$.

(e) (3 pts) By making use of $T(I, D)$ and (d), find an upper bound to the bit error probability (in terms of p).

Problem 4. (11 points)

Suppose $\phi(t)$ and $\xi(t)$ are two complex-valued low-pass waveforms, containing frequencies only in the frequency range $[-B, B]$. Suppose $f_0 > B$. Consider the following sequence of operations done on a complex number c to obtain a complex number y:

1. construct the real waveform $w(t) = \sqrt{2}\,\mathrm{Re}\{c\,\phi(t)\exp(\mathrm{j}2\pi f_0 t)\}$
2. construct the complex waveform $r(t) = \sqrt{2}\,w(t)\exp(-\mathrm{j}2\pi f_0 t)$
3. take the inner product of r and ξ to form $y = \langle r, \xi\rangle$.

(a) (2 pts) Show that $y = c\langle \phi, \xi\rangle$.

Consider now a transmitter that transforms the message i to a bandpass transmitted waveform $w_i(t)$ as follows (exactly in the way we discussed in class):

$$\big[i\big] \to \big[c_i \in \mathbb{C}^n\big] \to \Big[w_{i,E}(t) = \sum_j c_{ij}\phi_j(t)\Big] \to \big[w_i(t) = \sqrt{2}\mathrm{Re}\{w_{i,E}(t)\exp(\mathrm{j}2\pi f_0 t)\}\big].$$

Here $\phi_1, \dots, \phi_n$ are complex orthonormal, baseband waveforms (all supported in the frequency range $[-B, B]$), and $f_0 > B$.

The signal $w_i(t)$ is transmitted on an AWGN channel with noise intensity $N_0/2$, the received signal is $R(t)$.

The receiver operates as follows:

$$\big[R(t)\big] \to \big[R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_0 t)\big] \to \big[Y \in \mathbb{C}^n \text{ where } Y_j = \langle R, \xi_j\rangle\big] \to \big[\text{decision device}\big] \to \big[\hat{i}\big].$$

Note that the receiver forms Y using complex orthonormal baseband basis functions $\xi_1, \dots, \xi_n$ (all in the frequency range $[-B, B]$); had the ξ's been equal to ϕ's, then we would have our optimal receiver, with error probability $p_{\mathrm{opt}}(N_0)$.

(b) (3 pts) Find an $n \times n$ matrix A such that Y can be written in the form $Y = Ac_i + Z$, where Z is $\mathcal{N}_{\mathbb{C}}(0, N_0 I_n)$, i.e., $Z = Z_R + \mathrm{j}Z_I$ with $Z_R, Z_I \sim \mathcal{N}\big(0, \frac{N_0}{2} I_n\big)$ being independent.

Hint: Use (a) and express the entries A_{kj} of the matrix A in terms of the ϕ's and ξ's.

(c) (3 pts) Suppose $n = 2$ and $\langle\phi_1, \xi_1\rangle = 0$, $\langle\phi_1, \xi_2\rangle = \mathrm{j}$, $\langle\phi_2, \xi_1\rangle = 1$, $\langle\phi_2, \xi_2\rangle = 0$. Show how the decision device (that produces $\hat{i}$ from Y) can be implemented so that the error probability is equal to $p_{\mathrm{opt}}(N_0)$.

(d) (3 pts) Suppose $n = 2$ and $\langle\phi_1, \xi_1\rangle = \langle\phi_1, \xi_2\rangle = \langle\phi_2, \xi_2\rangle = 1/2$, $\langle\phi_2, \xi_1\rangle = -1/2$. Show that with the best possible decision device, the error probability will be $p_{\mathrm{opt}}(2N_0)$.