PROBLEM 1. Derive the power spectral density of the random process

$$X(t) = \sum_{i \in \mathbb{Z}} X_i \psi(t - iT - \Theta),$$

$\psi(t)$

where $\{X_i\}_{i \in \mathbb{Z}}$ is an i.i.d. sequence of uniformly distributed random variables taking values in $\{\pm\sqrt{\mathcal{E}}\}$, $\Theta$ is uniformly distributed in the interval $[0, T]$, and $\psi(t)$ is as shown in the plot (called Manchester pulse). The Manchester pulse guarantees that $X(t)$ has at least one transition per symbol, which facilitates the clock recovery at the receiver.
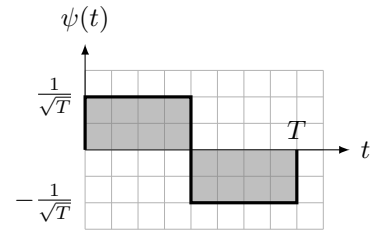
PROBLEM 2. Consider the random process

$$X(t) = \sum_{i \in \mathbb{Z}} X_i \sqrt{\mathcal{E}_s} \psi(t - iT_s - T_0),$$

where $T_s$ and $\mathcal{E}_s$ are fixed positive numbers, $\psi(t)$ is some unit-energy function, $T_0$ is a uniformly distributed random variable taking values in $[0, T_s)$, and $\{X_i\}_{i \in \mathbb{Z}}$ is the output of the convolutional encoder described by

$$X_{2n} = B_n B_{n-2}, \qquad X_{2n+1} = B_n B_{n-1} B_{n-2},$$

with i.i.d. input sequence $\{B_i\}_{i \in \mathbb{Z}}$ taking values in $\{\pm 1\}$.

(a) Express the power spectral density of $X(t)$ for a general $\psi(t)$.

(b) Plot the power spectral density of $X(t)$ assuming that $\psi(t)$ is a unit-norm rectangular pulse of width $T_s$.

PROBLEM 3. From the decoder's point of view, inter-symbol interference (ISI) can be modeled as follows:

$$Y_i = X_i + Z_i$$

$$X_i = \sum_{j=0}^{L} B_{i-j} h_j, \quad i \in \mathbb{N} \tag{$*$}$$

where $B_i$ is the $i$th information bit, $h_0, \ldots, h_L$ are coefficients that describe the inter-symbol interference, and $Z_i$ is zero-mean, Gaussian, of variance $\sigma^2$, and statistically independent of everything else. Relationship $(*)$ can be described by a trellis, and the ML decision rule can be implemented by the Viterbi algorithm.

(a) Draw the trellis that describes all sequences of the form $X_1, \ldots, X_6$ resulting from information sequences of the form $B_1, \ldots, B_5, 0$, $B_i \in \{0, 1\}$, assuming

$$h_i = \begin{cases} 1, & i = 0 \\ -2, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

To determine the initial state, you may assume that the preceding information sequence terminated with 0. Label the trellis edges with the input/output symbols.

(b) Specify a metric $f(x_1, \ldots, x_6) = \sum_{i=1}^{6} f(x_i, y_i)$ whose minimization or maximization with respect to the valid $x_1, \ldots, x_6$ leads to a maximum likelihood decision. Specify if your metric needs to be minimized or maximized.

(c) Assume $y_1, \ldots, y_6 = \{2, 0, -1, 1, 0, -1\}$. Find the maximum likelihood estimate of the information sequence $B_1, \ldots, B_5$.

PROBLEM 4. An output sequence $x_1, \ldots, x_{10}$ from the convolutional encoder shown below is transmitted over the discrete-time AWGN channel. The initial and final state of the encoder is $(1, 1)$. Using the Viterbi algorithm, find the maximum likelihood information sequence $\hat{b}_1, \ldots, \hat{b}_4, 1, 1$, knowing that $b_1, \ldots, b_4$ are drawn independently and uniformly from $\{\pm 1\}$ and that the channel output $y_1, \ldots, y_{10} = \{1, 2, -1, 4, -2, 1, 1, -3, -1, -2\}$. (It is for convenience that we are choosing integers rather than real numbers.)
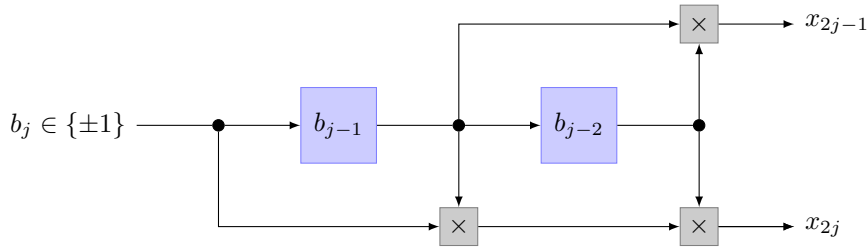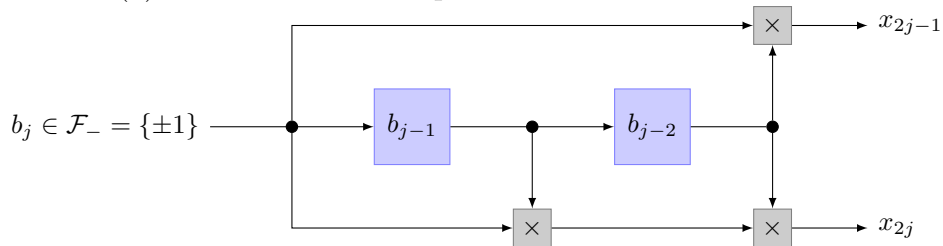


PROBLEM 5. Consider the following two encoders where the map $T : \mathcal{F}_0 \rightarrow \mathcal{F}_-$ sends 0 to 1 and 1 to $-1$. Show that the two encoders produce the same output when their inputs are related by $b_j = T(\bar{b}_j)$.

*Hint:* For $a, b \in \mathcal{F}_0$, $T(a + b) = T(a) \times T(b)$, where addition is modulo 2 and multiplication is over $\mathbb{R}$.
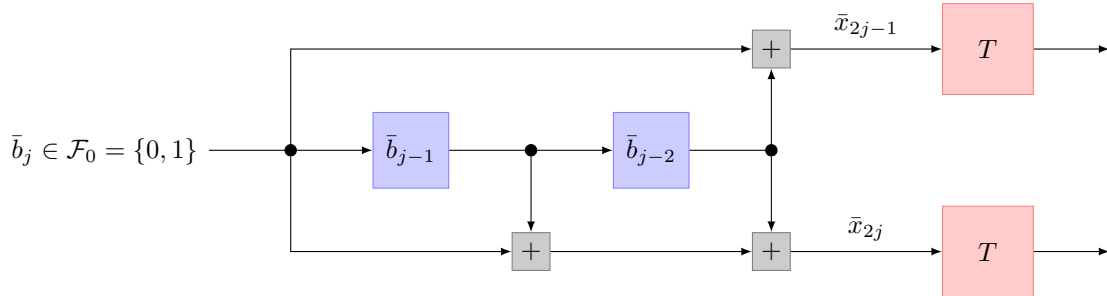


(a) Conventional description. Addition is modulo 2.



(b) Description used in the book. Multiplication is over $\mathbb{R}$.

*Comment:* The encoder of (b) is linear over the field $\mathcal{F}_-$, whereas the encoder of (a) is linear over $\mathcal{F}_0$ only if we omit the output map $T$. The comparison of the two figures should explain why in this chapter we have opted for the description of (b) even though the standard description of a convolutional encoder is as in (a).