ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 25

Solutions to Problem Set 10

Principles of Digital Communications May 10, 2024

SOLUTION 1. In Section 5.3, it is shown that the power spectral density is

$$S_X(f) = \frac{|\psi_{\mathcal{F}}(f)|^2}{T} \sum_{k} K_X[k] \exp(-j2\pi k f T),$$

where $K_X[k]$ is the auto-covariance of X_i and $\psi_{\mathcal{F}}(f)$ is the Fourier transform of $\psi(t)$. Because $\{X_i\}_{i=-\infty}^{\infty}$ are i.i.d. and have zero-mean,

$$K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \mathcal{E}1\{k=0\},$$

SO

$$S_X(f) = \mathcal{E} \frac{|\psi_{\mathcal{F}}(f)|^2}{T}.$$

Moreover,

$$\psi_{\mathcal{F}}(f) = \int_{-\infty}^{\infty} \psi(t)e^{-j2\pi ft} dt$$

$$= \frac{1}{\sqrt{T}} \int_{0}^{\frac{T}{2}} e^{-j2\pi ft} dt - \frac{1}{\sqrt{T}} \int_{\frac{T}{2}}^{T} e^{-j2\pi ft} dt$$

$$= \frac{j}{2\pi f\sqrt{T}} \left(e^{-j2\pi f\frac{T}{2}} - 1 - e^{-j2\pi fT} + e^{-j2\pi f\frac{T}{2}} \right)$$

$$= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} \left(2 - e^{j2\pi f\frac{T}{2}} - e^{-j2\pi f\frac{T}{2}} \right)$$

$$= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} (2 - 2\cos(\pi fT))$$

$$= \frac{j}{2\pi f\sqrt{T}} e^{-j2\pi f\frac{T}{2}} 4\sin^{2}\left(\pi f\frac{T}{2}\right).$$

Therefore,

$$S_X(f) = \mathcal{E} \frac{16 \sin^4(\pi f \frac{T}{2})}{4\pi^2 f^2 T^2}$$
$$= \mathcal{E} \left(\pi f \frac{T}{2}\right)^2 \operatorname{sinc}^4\left(f \frac{T}{2}\right).$$

SOLUTION 2.

(a) When i = j, $\mathbb{E}[X_i X_j]$ equals

$$\mathbb{E}[X_i^2] = \mathbb{E}[1] = 1.$$

Remember that the B_i are i.i.d. Bernoulli($\frac{1}{2}$) random variables. Hence, we find immediately

$$K_X[1] = \mathbb{E}[X_{2n}X_{2n+1}] = \mathbb{E}[B_nB_{n-2}B_nB_{n-1}B_{n-2}]$$

= $\mathbb{E}[B_n^2B_{n-1}B_{n-2}^2]$
= $\mathbb{E}[B_{n-1}] = 0,$

and also

$$K_X[2] = \mathbb{E}[X_{2n}X_{2n+2}] = \mathbb{E}[B_nB_{n-2}B_{n+1}B_{n-1}]$$

= $\mathbb{E}[B_n] \mathbb{E}[B_{n-2}] \mathbb{E}[B_{n+1}] \mathbb{E}[B_{n-1}] = 0.$

By continuing this argument we find

$$K_X[i] = 1\{i = 0\}.$$

Hence,

$$S_X(f) = \frac{\mathcal{E}_s}{T_s} |\psi_{\mathcal{F}}(f)|^2.$$

This means that by choosing $\psi(t)$ appropriately, we can control the bandwidth consumption of our communications scheme.

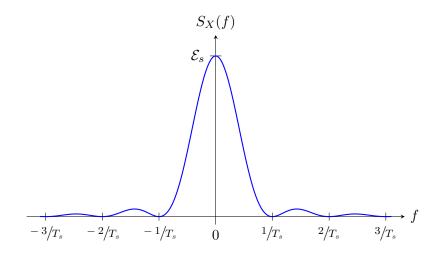
(b) We know that

$$|\psi_{\mathcal{F}}(f)|^2 = T_s \operatorname{sinc}^2(T_s f).$$

It follows that

$$S_X(f) = \mathcal{E}_s \operatorname{sinc}^2(T_s f).$$

A plot of $S_X(f)$ is shown below:

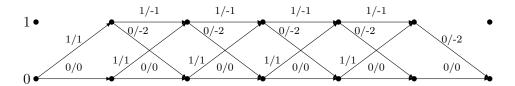


SOLUTION 3.

(a)

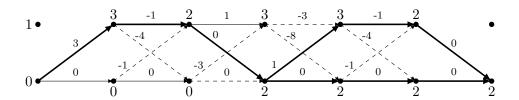
$$X_i = B_i - 2B_{i-1}$$

From this, we can draw the following trellis:



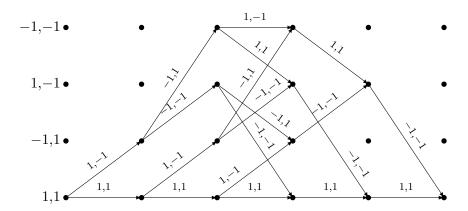
(b) We have Y = X + Z, where $Z = (Z_1, \ldots, Z_6)$ is a sequence of i.i.d. components with $Z_i \sim \mathcal{N}(0, \sigma^2)$. Our maximum likelihood decoder is a minimum distance decoder. We have to minimize $||y - x||^2$ or equivalently, maximize $2\langle y, x \rangle - ||x||^2$. We thus have $f(x, y) = \sum_{i=1}^{6} (2y_i x_i - x_i^2)$ whose maximization with respect to x leads to a maximum likelihood decision on X.

(c) We label our trellis with the edge metric $2y_ix_i - x_i^2$ and then trace back the decoding path.

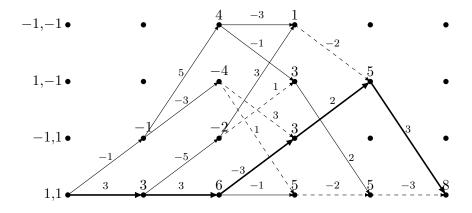


We see that the two sequences 1,1,0,0,0 and 1,1,0,1,1 are equally likely, so the decoder would choose either of the two.

SOLUTION 4. The trellis representing the encoder is shown below:



We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.



From the figure we can read the decoded sequence 1, 1, -1, 1, 1.

SOLUTION 5. The output of encoder (a) is

$$\begin{split} T(\bar{x}_{2j-1}) &= T(\bar{b}_j + \bar{b}_{j-2}) = T(\bar{b}_j) T(\bar{b}_{j-2}) = b_j b_{j-2} \\ T(\bar{x}_{2j}) &= T(\bar{b}_j + \bar{b}_{j-1} + \bar{b}_{j-2}) = T(\bar{b}_j) T(\bar{b}_{j-1}) T(\bar{b}_{j-2}) = b_j b_{j-1} b_{j-2}, \end{split}$$

which is identical to the output of encoder (b).