ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 16 Principles of Digital Communications
Midterm exam Apr. 19, 2024

4 problems, 43 points, 165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. (7 points)

Suppose $Z = (Z_1, Z_2)$ is uniformly distributed on the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$. In a binary hypothesis problem the observation Y is given by

$$Y = \begin{cases} Z & \text{if } H = 0, \\ \beta Z & \text{if } H = 1, \end{cases}$$

where $\beta > 1$ is a known constant. Let $p_0 = \Pr(H = 0)$ and $p_1 = 1 - p_0 = \Pr(H = 1)$.

(a) (3 pts) Find the MAP decision rule $\hat{H}_{\text{MAP}}(y)$.
Hint: If X is uniformly distributed on a set $A \subset \mathbb{R}^2$, then $f_X(x) = \frac{1}{\text{Area}(A)} \mathbb{1}\{x \in A\}$.

(b) (2 pts) Are there values of p_0 for which the MAP rule does not depend on y? If so, find them.

(c) (2 pts) Assume $p_0 = 1/2$. Find $\Pr(\text{error}|H = 0)$ and $\Pr(\text{error}|H = 1)$.

Problem 2. (12 points)

Suppose $Z = [Z_1, Z_2, Z_3]^T \sim \mathcal{N}(0, K)$, with

$$K = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(a) (2 pts) Show that $Z_1 + Z_2 + Z_3 = 0$ with probability 1.
Hint: $E[X^2] = 0$ implies that $X = 0$ with probability 1.

(b) (3 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1/\sqrt{3} & 2/\sqrt{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}$, and let $U = AZ$. What is the covariance matrix of U?
Hint: Show that U_1 and U_2 are independent, and use (a).

Let $c_1 = [1, 2, 3]^T$ and $c_2 = [5, 1, 0]^T$ be the codewords of a communication system with two equally likely messages, and suppose $Y = c_i + Z$ (with Z as above) be the receiver's observation if message i is sent.

(c) (2 pts) Let $\tilde{Y} = AY$. (Note that A is an invertible matrix, so $\tilde{Y}$ is equivalent to Y.) Show that $(\tilde{Y}_1, \tilde{Y}_2)$ is a sufficient statistic.

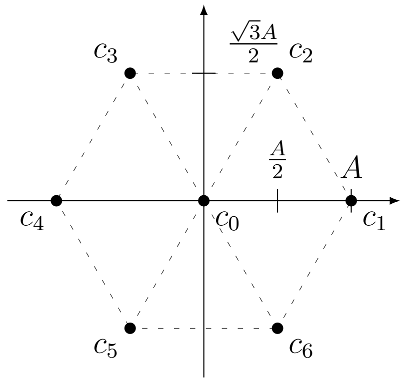
(d) (3 pts) Find the probability of error of the MAP decision rule for the communication system above.

Suppose we replace c_1 and c_2 above with $c_1 = [0, 0, 1]^T$ and $c_2 = [0, 0, -1]^T$. The observation Y is still $c_i + Z$, and $\tilde{Y} = AY$.

(e) (2 pts) What is the probability of error of the MAP decision rule for this new system? Is $(\tilde{Y}_1, \tilde{Y}_2)$ still a sufficient statistic? (Explain).

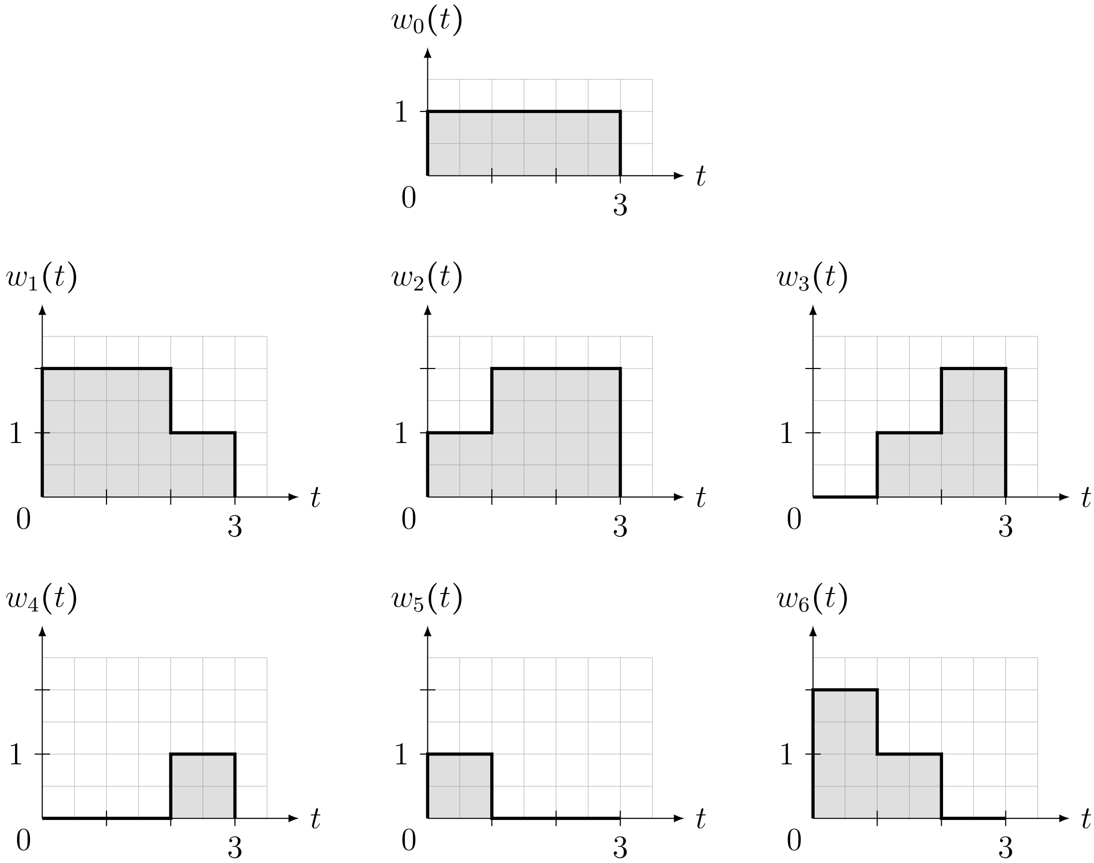
Problem 3. (11 points)

Consider the constellation with seven codewords $\{c_i\}_{i=0}^{6}$ as given in the diagram. Assume that the seven messages are equally likely, and let $e_{\text{hex}}(A)$ be the error probability of the MAP decoder that observes $Y = c_i + Z$, where $Z \sim \mathcal{N}(0, I_2)$.



(a) (3 pts) Show that $e_{\text{hex}}(A)$ is upper bounded by $\Pr\left(Z_1^2 + Z_2^2 \geq A^2/4\right)$.
Hint: You may find it helpful to draw the decision regions.

Now consider the waveforms $\{w_i\}_{i=0}^{6}$ as shown below.



(b) (2 pts) Assume that all messages are equally likely. Find a translation of this waveform set to minimize the average energy. Let the new waveforms be $\{\tilde{w}_i\}_{i=0}^{6}$.

(c) (2 pts) Show that $\tilde{w}_1 + \tilde{w}_4 = \tilde{w}_2 + \tilde{w}_5 = \tilde{w}_3 + \tilde{w}_6 = 0$, and that $\|\tilde{w}_1\| = \|\tilde{w}_2\| = \|\tilde{w}_3\|$.

(d) (2 pts) Find the inner products $\langle \tilde{w}_1, \tilde{w}_2 \rangle$, $\langle \tilde{w}_1, \tilde{w}_3 \rangle$, and $\langle \tilde{w}_2, \tilde{w}_3 \rangle$.

(e) (2 pts) Consider a communication system which uses the waveforms $\{w_i\}_{i=0}^{6}$ to communicate over a white Gaussian noise channel with intensity $\frac{N_0}{2}$. Express the optimal error probability of this system *in terms of* $e_{\text{hex}}(\cdot)$.
Hint: No lengthy computations needed.

PROBLEM 4. (13 points)

Let $e_1, \dots, e_n$ denote the standard basis for $\mathbb{R}^n$, i.e., $e_1 = [1, 0, \dots, 0]^T$, $e_2 = [0, 1, 0, \dots, 0]^T$, $\dots$, $e_n = [0, \dots, 0, 1]^T$.
We have a communication system with $m = 2n$ codewords $c_1, \dots, c_{2n}$ in $\mathbb{R}^n$, with $c_i = Ae_i$ for $i = 1, \dots, n$, and $c_i = -c_{i-n}$ for $i = n+1, \dots, 2n$. Here $A > 0$ is a positive constant.
The m messages are equally likely, and the receiver's observation is given by $Y = c_i + Z$ if message i is sent, where $Z \sim \mathcal{N}(0, \sigma^2 I_n)$.

(a) (2 pts) Find the average energy $\mathcal{E}$ and the average energy per bit $\mathcal{E}_b$ of the signal constellation above.

(b) (2 pts) Consider the decision method that first computes $i_0 = \arg\max_{i=1,\dots,n} |y_i|$, and sets $\hat{H}(y) = \begin{cases} i_0 & \text{if } y_{i_0} > 0, \\ n + i_0 & \text{else.} \end{cases}$
Is this rule optimal? (Explain your answer.)

(c) (2 pts) Upper bound the probability of error using the union bound.
Hint: For each codeword c_i, the codeword $-c_i$ is at distance $2A$ from it; what is the distance between c_i and the other $2n - 2$ codewords?

(d) (2 pts) Show that if $\mathcal{E}_b/\sigma^2 > 4 \ln 2$, then the probability of error of this communication system approaches zero as n gets large.
Hint: Use (c), that $Q(\sqrt{x}) \le \exp(-x/2)$, and note that $(m-1)\exp(-\alpha \log_2(m))$ tends to zero as m gets large if $\alpha > \ln 2$.

(e) (2 pts) Show that for $i = 1, \dots, n$,

$$\ln \frac{p_{H|Y}(i|y)}{p_{H|Y}(n+i|y)} = 2Ay_i/\sigma^2.$$

(f) (3 pts) Suppose $T = t(Y)$ where $t(\cdot)$ is a deterministic function. Suppose that there exist y and $\tilde{y}$, with $y \ne \tilde{y}$ and $t(y) = t(\tilde{y})$. Can T be a sufficient statistic? Explain.
Hint: Use (e).