ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 2 Information Theory and Coding
Homework 1 Sep. 9, 2024

PROBLEM 1. Three events E_1, E_2 and E_3, defined on the same probability space, have probabilities $P(E_1) = P(E_2) = P(E_3) = 1/4$. Let E_0 be the event that one or more of the events E_1, E_2, E_3 occurs.

(a) Find $P(E_0)$ when:

  (1) The events E_1, E_2 and E_3 are disjoint.

  (2) The events E_1, E_2 and E_3 are independent.

  (3) The events E_1, E_2 and E_3 are in fact three names for the same event.

(b) Find the maximum value $P(E_0)$ can take when:

  (1) Nothing is known about the independence or disjointness of E_1, E_2, E_3.

  (2) It is known that E_1, E_2 and E_3 are *pairwise independent*, i.e., that the probability of realizing both E_i and E_j is $P(E_i)P(E_j)$, $1 \le i \ne j \le 3$, but nothing is known about the probability of realizing all three events together.

(c) Suppose now that events E_1, E_2 and E_3 all have probability p, that they are pairwise independent, and that E_0 has probability 1. Show that p has to be at least $1/2$.

PROBLEM 2. A child is playing a game and tosses a fair die until the first 6 comes. Here, the number of tosses is a random variable denoted by N_1. N_1 takes values in $\{1, 2, \ldots\}$

(a) Find $P(N_1 = k), k \in \{1, 2, \ldots\}$

(b) Find $E[N_1]$. (*Hint:* $\sum_{k=1}^{\infty} x^{k-1}k = 1/(1-x)^2$)

(c) The child tries to make the game a little bit longer. Now, he stops the game when he gets the m^{th} 6. For example, when $m = 2$, he stops when the observed sequence is $1, 4, 2, 3, 6, 2, 3, 4, 6$. Denote the new random variable by $\tilde{N}$, where $\tilde{N}$ takes values in $\{m, m+1, \ldots\}$. Repeat (a) and (b) for this case.

(d) This child has a older brother and he has a loaded dice with identical appearance and $P(\text{Top face shows 6}) = 1/6^5$. He takes the fair dice from his little brother and puts both die in a bag. The child then chooses a die at random. Suppose that he observes the first 6 at k^{th} outcome. Based on this observation, what is the posterior probability that the die is fair? For which range of k is $P(\text{Fair} \mid N_1 = k) < P(\text{Loaded} \mid N_1 = k)$?

PROBLEM 3. Suppose the random variables A, B, C, D form a Markov chain: $A \multimap B \multimap C \multimap D$.

(a) Is $A \multimap B \multimap C$?

(b) Is $B \multimap C \multimap D$?

(c) Is $A \minuso (B, C) \minuso D$?

PROBLEM 4. Suppose the random variables A, B, C, D satisfy $A \minuso B \minuso C$, and $B \minuso C \minuso D$. Does it follow from these that $A \minuso B \minuso C \minuso D$?

PROBLEM 5. Let X and Y be two random variables.

(a) Prove that the expectation of the sum of X and Y, $E[X + Y]$, is equal to the sum of the expectations, $E[X] + E[Y]$.

(b) Prove that if X and Y are independent, then X and Y are also uncorrelated (by definition X and Y are uncorrelated if $E[XY] = E[X]E[Y]$). Find an example in which X and Y are dependent yet uncorrelated.

(c) Prove that if X and Y are independent, then the variance of the sum $X + Y$ is equal to the sum of variances. Is this relationship valid if X and Y are uncorrelated but not independent?

PROBLEM 6. After summer, the winter tyres of a car (with four wheels) are to be put back. However, the owner has forgotten which tyre goes to which wheel, and the tyres are installed 'randomly', each of the $4! = 24$ permutations being equally likely.

(a) What is the probability that tyre 1 is installed in its original position?

(b) What is the probability that all the tyres are installed in their original positions?

(c) What is the expected number of tyres that are installed in their original positions?

(d) Redo the above for a vehicle with n wheels.

(e) (Harder.) What is the probability that none of the wheels are installed in their original positions.

PROBLEM 7. We construct an 'inventory' by drawing n independent samples from a distribution p. Let $X_1, \ldots, X_n$ be the random variables that represent the drawings.

Suppose X is drawn from distribution p, independent of $X_1, \ldots, X_n$.

(a) What is the probability that X does not appear in the inventory?

(b) Redo (a) for the special case when p is the uniform distribution over n items.

(c) What happens to the probability in (b) when n gets large?