

# ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

## Handout 13

### Homework 6

Information Theory and Coding

Oct. 14, 2024

PROBLEM 1. Suppose  $L : \mathbb{R}^K \rightarrow \mathbb{R}^N$  is a linear function and  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  is a concave function. Show that  $f : \mathbb{R}^K \rightarrow \mathbb{R}$  defined as  $f(x) = g(L(x))$  is concave.

PROBLEM 2. From the notes on Lempel-Ziv algorithm, we know that the total length  $n$  of  $c$  distinct binary strings satisfies

$$n > c \log_2(c/8)$$

The same technique, when applied to the non-binary strings yields

$$n > c \log_K(c/K^3)$$

where  $K$  is the size of the alphabet the letters of the string belong to. This inequality lower bounds  $n$  in terms of  $c$ . We will now show that  $n$  can also be upper bounded in terms of  $c$ .

- (a) Show that, if  $n \geq \frac{1}{2}m(m-1)$ , then  $c \geq m$ .
- (b) Find a sequence for which the bound in (a) is met with equality.
- (c) Show now that  $n < \frac{1}{2}c(c+1)$ .

PROBLEM 3. Let  $X$  be the channel input. Assume that the channel output  $Y$  is passed through a data processor in such a way that no information is lost. That is,

$$I(X; Y) = I(X; Z)$$

where  $Z$  is the processor output. Find an example where  $H(Y) > H(Z)$  and find an example where  $H(Y) < H(Z)$ .

*Hint:* The data processor does not have to be deterministic

PROBLEM 4. A “ $K$ -ary erasure channel with erasure probability  $p$ ” is described as follows: the input  $U$  belongs to the alphabet  $\{1, \dots, K\}$ , the output  $V$  belongs to the alphabet  $\{1, \dots, K\} \cup \{?\}$ , and if  $u$  is the input, the output  $V$  equals  $u$  with probability  $1 - p$ , and equals  $?$  with probability  $p$ . Note that  $\Pr(V = ?) = p$  regardless of the input distribution.

- (a) Show that  $\Pr(U = u | V = ?) = p_U(u)$ .
- (b) Show that  $I(U; V) = (1 - p)H(U)$ .
- (c) Find the capacity of this channel and the input distribution that maximizes the mutual information.

PROBLEM 5. We are given a memoryless stationary binary symmetric channel  $\text{BSC}(\epsilon)$ . Namely, if  $X_1, \dots, X_n \in \{0, 1\}$  are the input of this channel and  $Y_1, \dots, Y_n \in \{0, 1\}$  are the output, we have:

$$P(Y_i | X_i, X^{i-1}, Y^{i-1}) = P(Y_i | X_i) = \begin{cases} 1 - \epsilon & \text{if } Y_i = X_i, \\ \epsilon & \text{otherwise.} \end{cases}$$

Let  $W$  be a random variable that is uniform in  $\{0, 1\}$  and consider a communication system with feedback which transmits the value of  $W$  to the receiver as follows:

- At time  $t = 1$ , the transmitter sends  $X_1 = W$  through the channel.
  - At time  $t = i + 1 \leq n$ , the transmitter gets the value of  $Y_i$  from the feedback and sends  $X_{i+1} = Y_i$  through the channel.
- (a) Give the capacity  $C$  of the channel in terms of  $\epsilon$ , and show that  $C = 0$  when  $\epsilon = \frac{1}{2}$ .
- (b) Show that if  $\epsilon = \frac{1}{2}$ ,  $I(X^n; Y^n) = n - 1$ . This means that  $I(X^n; Y^n) \leq nC$  does not hold for this system.
- (c) Show that although  $I(X^n; Y^n) > nC$  when  $\epsilon = \frac{1}{2}$ , we still have  $I(W; Y^n) \leq nC$ .

Note that since  $W$  is the useful information that is being transmitted, it is the value of  $I(W; Y^n)$  that we are interested in when we want to compute the amount of information that is shared with the receiver.