PROBLEM 1. Random variables X and Y are correlated Gaussian variables:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} : K = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\right).$$

Find $I(X;Y)$.

PROBLEM 2. Consider an additive noise channel with input $x \in \mathbb{R}$, and output

$$Y = x + Z$$

where Z is a real random variable independent of the input x, has zero mean and variance equal to σ^2.

In this problem we prove in a different way from the lecture that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance. Let $\mathcal{N}_{\sigma^2}$ denote the Gaussian density with zero mean and variance σ^2.

(a) Denote the input probability density by p_X. Verify that

$$I(X;Y) = \iint p_X(x)p_Z(y-x)\ln\frac{p_Z(y-x)}{p_Y(y)}\,dxdy \quad \text{nats.}$$

where p_Y is the density of the output when the input has density p_X.

(b) Now set $p_X = \mathcal{N}_P$. Verify that

$$\tfrac{1}{2}\ln\bigl(1 + P/\sigma^2\bigr) = \iint p_X(x)p_Z(y-x)\ln\frac{\mathcal{N}_{\sigma^2}(y-x)}{\mathcal{N}_{P+\sigma^2}(y)}\,dxdy.$$

(c) Still with $p_X = \mathcal{N}_P$, show that

$$\tfrac{1}{2}\ln(1 + P/\sigma^2) - I(X;Y) \le 0.$$

[Hint: use (a) and (b) and $\ln t \le t - 1$.]

(d) Show that an additive noise channel with noise variance σ^2 and input power P has capacity at least $\frac{1}{2}\log_2(1 + P/\sigma^2)$ bits per channel use. Conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

PROBLEM 3. A discrete memoryless channel has three input symbols: $\{-1, 0, 1\}$, and two output symbols: $\{1, -1\}$. The transition probabilities are

$$p(-1|-1) = p(1|1) = 1, \quad p(1|0) = p(-1|0) = 0.5.$$

Find the capacity of this channel with cost constraint β, if the cost function is $b(x) = x^2$.

PROBLEM 4. Consider a vector Gaussian channel described as follows:

$$Y_1 = x + Z_1$$
$$Y_2 = Z_2$$

where x is the input to the channel constrained in power to P; Z_1 and Z_2 are jointly Gaussian random variables with $E[Z_1] = E[Z_2] = 0$, $E[Z_1^2] = E[Z_2^2] = \sigma^2$ and $E[Z_1 Z_2] = \rho\sigma^2$, with $\rho \in [-1, 1]$, and independent of the channel input.

(a) Consider a receiver that discards Y_2 and decodes the message based only on Y_1. What rates are achievable with such a receiver?

(b) Consider a receiver that forms $Y = Y_1 - \rho Y_2$, and decodes the message based only on Y. What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

PROBLEM 5. Suppose $(X_1, \dots, X_n, Y_1, \dots, Y_m)$ is an $n + m$ dimensional Gaussian random vector with covariance matrix K. Partition the $(n + m) \times (n + m)$ matrix K as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where K_{11} is $n \times n$ and K_{22} is $m \times m$.

(a) Express $h(X_1, \dots, X_n)$, $h(Y_1, \dots, Y_n)$ and $h(X_1, \dots, X_n, Y_1, \dots, Y_m)$ in terms of the matrices above.

(b) Show if the matrix $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ is positive definite, then $\det(A) \le \det(A_{11}) \det(A_{22})$. [Hint: for any positive definite matrix A, $f(x) = \det(2\pi A)^{-1/2} \exp(-\frac{1}{2} x^T A^{-1} x)$ is a probability density.]