ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 23 Information Theory and Coding
Homework 10 Nov. 25, 2024

PROBLEM 1. Suppose U is $\{0,1\}$ valued with $\mathbb{P}(U=0)=\mathbb{P}(U=1)=1/2$. Suppose we have a distortion measure d given by

$$d(u,v)=\begin{cases}0, & u=v\\ 1, & (u,v)=(1,0)\\ \infty, & (u,v)=(0,1)\end{cases}$$

i.e., we never want to represent a 0 with a 1. Find $R(D)$.

PROBLEM 2. Suppose $\mathcal{U}=\mathcal{V}$ are additive groups with group operation $\oplus$. (E.g., $\mathcal{U}=\mathcal{V}=\{0,\dots,K-1\}$, with modulo K addition.) Suppose the distortion measure $d(u,v)$ depends only on the difference between u and v and is given by $g(u\ominus v)$. Let $\phi(D)$ denote $\max\{H(Z):E[g(Z)]\le D\}$.

a) Show that $\phi(D)$ is concave.

b) Let (U,V) be such that $E[d(U,V)]\le D$. Show that $I(U;V)\ge H(U)-\phi(D)$ by justifying

$$I(U;V)=H(U)-H(U|V)=H(U)-H(U\ominus V|V)\ge H(U)-H(U\ominus V)\ge H(U)-\phi(D).$$

c) Show that $R(D)\ge H(U)-\phi(D)$.

d) Assume now that U is uniform on $\mathcal{U}$. Show that $R(D)=H(U)-\phi(D)$.

PROBLEM 3. Suppose $\mathcal{U}=\mathcal{V}=\mathbb{R}$, the set of real numbers, and $d(u,v)=(u-v)^2$.

(a) Show that for any U with variance σ^2, $R(D)$ satisfies

$$h(U)-\tfrac{1}{2}\log(2\pi eD)\le R(D).$$

(b) Show that $R(D)$ does not depend on the mean of U.

Now, assume without loss of generality that U is zero-mean. Suppose we have access to a noisy observation V of U through the channel $U+Z=V$, where $Z\sim\mathcal{N}(0,\sigma_Z^2)$ and independent of U. We reconstruct U via a linear estimator $\hat{U}=aV+b$.

(c) Show that $a=\frac{\sigma^2}{\sigma^2+\sigma_Z^2}$ and $b=0$ minimizes $E[(U-\hat{U})^2]$ and for such choice of a, b, $E[(U-\hat{U})^2]=\sigma^2\frac{\sigma_Z^2}{\sigma^2+\sigma_Z^2}$.

(d) For the channel above, show that

$$I(U;V)\le\tfrac{1}{2}\log\Big(1+\tfrac{\sigma^2}{\sigma_Z^2}\Big)$$

(e) Show that for $D\le\sigma^2$

$$R(D)\le\tfrac{1}{2}\log(\sigma^2/D).$$

[Hint: Use the channel above for a candidate $p_{V|U}$.]

PROBLEM 4. Given finite alphabets $\mathcal{X}$ and $\mathcal{Y}$, a distribution p_{XY}, $0 < \epsilon < \epsilon'$, and a sequence $x^n \in T(n, p_X, \epsilon)$, consider a random vector Y^n with independent components with $\Pr(Y_i = y) = p_{Y|X}(y|x_i)$.

For $x \in \mathcal{X}$, let $J(x) = \{i : x_i = x\}$. For an $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, let $N(x, y) = \sum_i \mathbb{1}\big\{(x_i, Y_i) = (x, y)\big\} = \sum_{i \in J(x)} \mathbb{1}\{Y_i = y\}$.

(a) Show that for each x and y, $np(x, y)(1 - \epsilon) \leq E[N(x, y)] \leq np(x, y)(1 + \epsilon)$, and $\mathrm{Var}(N(x, y))$ is at most n. [Hint: don't forget that x^n is in $T(n, p_X, \epsilon)$.]

(b) Show that for each x and y, both $\Pr\big(N(x, y) < np(x, y)(1 - \epsilon')\big)$ and $\Pr\big(N(x, y) > np(x, y)(1 + \epsilon')\big)$ approach to zero as n gets large. Would this be true if we had not assumed $\epsilon < \epsilon'$?

(c) Using (a) and (b) show that $\Pr\big((x^n, Y^n) \notin T(n, p_{XY}, \epsilon')\big)$ approaches 0 as gets large.

(d) Suppose now we have a distribution $p(u, x, y)$ where $p(y|ux) = p(y|x)$. [In other words, U, X, Y form a Markov chain.] Suppose (u^n, x^n) is in $T(n, p_{UX}, \epsilon)$, and Y^n has independent components as above. What can we say about $\Pr\big((u^n, x^n, Y^n) \in T(n, p_{UXY}, \epsilon')\big)$?