ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 34 Information Theory and Coding
Final exam Jan. 13, 2025

4 problems, 34 points, 180 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

(All logarithms are taken to the base 2.)

PROBLEM 1. (9 points)

(a) (2 pts) Suppose X and Y are random variables and suppose that X is uniformly distributed on a finite set of values. Let $p = \Pr(X = Y)$. Show that $I(X;Y) \geq pH(X) - 1$.

Hint: Upper bound $H(X|Y)$ by Fano's inequality. (Note that in this exercise Y need not be discrete.)

Consider now a discrete memoryless channel for which the input $x \in [0, 1]$ and the output $Y \in [0, 1]$ are related via $Y = \min(x, Z)$ where Z is a random variable uniformly distributed on $[0, 1]$.

(b) (2 pts) Fix a positive integer k and a positive number c with $0 < c \leq 1$. Suppose the input X is uniformly distributed on $\left\{\frac{ci}{k} : i = 0, \ldots, k\right\}$. Show that $I(X;Y) \geq (1 - c)\log(1 + k) - 1$.

(c) (2 pts) Find the capacity of this channel.

(d) (3 pts) Fix $0 \leq a < b \leq 1$. Suppose now the input x is constrained to be in the interval $[a, b]$. Find the capacity of the channel under this constraint.

Hint: Pick $c < b - a$, and let X be uniformly distributed on $\left\{a + \frac{ci}{k} : i = 0, \ldots, k\right\}$.

Problem 2. (7 points)

Recall that the Hamming weight $w_H(x^n)$ of a binary vector x^n is the number of 1's that occur in x^n, i.e., $w_H(x^n) = \sum_{i=1}^n \mathbb{1}(x_i = 1)$. Suppose $X^n \in \{0,1\}^n$ is a random binary vector, with $\frac{1}{n}\mathbb{E}[w_H(X^n)] = p$.

(a) (1 pts) Let $p_i = \Pr(X_i = 1)$. How are $p_1, \ldots, p_n$ and p related?

(b) (1 pts) How are $H(X_i)$ and p_i related?

(c) (3 pts) Show that $\frac{1}{n}H(X^n) \leq h_2(p)$.

(d) (2 pts) Let $B_n(r)$ be the Hamming ball of radius r around 0^n, i.e., $B_n(r) = \{x^n \in \{0,1\}^n : w_H(x^n) \leq r\}$. For $r \leq \frac{n}{2}$, show that $\frac{1}{n}\log|B_n(r)| \leq h_2(\frac{r}{n})$.
Hint: Let X^n be uniformly distributed on $B_n(r)$.

PROBLEM 3. (7 points)

Suppose X is an *integer valued* random variable, Z is uniformly distributed on the interval $[0,1]$ and is independent of X, and $Y = X + Z$.

(a) (2 pts) How is $H(X)$ and $h(Y)$ related?

(b) (2 pts) Show that $\mathrm{Var}(Y) = \mathrm{Var}(X) + \frac{1}{12}$.

(c) (1 pts) Show that $H(X) \leq \frac{1}{2}\log\Big(2\pi e\left(\mathrm{Var}(X) + \frac{1}{12}\right)\Big)$.

(d) (2 pts) Suppose $S_n = \sum_{i=1}^{n} B_i$ where B_i are i.i.d. $\mathrm{Bern}(\frac{1}{2})$. Show that $H(S_n) \leq \frac{1}{2}\log\left(2\pi e(\frac{n}{4} + \frac{1}{12})\right) \leq \frac{1}{2}\log(n+1) + \frac{1}{2}\log\left(\frac{\pi e}{2}\right)$

Problem 4. (11 points)

We are given a binary input channel $W : \mathbb{F}_2 \to \mathcal{Y}$. Let $Q(W) = \sum_y \sqrt{W(y|0)W(y|1)}$.

(a) (1 pts) Find $Q(\text{BEC}(p))$.

(b) (3 pts) Suppose the channel input X is equally likely to be 0 or 1, and upon observing the channel output $Y = y$, we estimate the value of X as $\hat{x}(y) = 0$ if $W(y|0) > W(y|1)$; 1 else. Show that $\Pr(\hat{x}(Y) \neq X) \leq Q(W)$.
Hint: First condition on $X = 0$. In this case $\mathbb{1}\{\hat{x}(y) \neq X\} \leq \sqrt{\frac{W(y|1)}{W(y|0)}}$.

Recall the polar construction which, from two instances of the channel W synthesized the channels $W^- : \mathbb{F}_2 \to \mathcal{Y}^2$ and $W^+ : \mathbb{F}_2 \to \mathcal{Y}^2 \times \mathbb{F}_2$ with

$$W^-(y_1y_2|u_1) = \frac{W(y_1|u_1)W(y_2|0) + W(y_1|u_1 \oplus 1)W(y_2|1)}{2}$$

and

$$W^+(y_1y_2u_1|u_2) = \frac{1}{2}W(y_1|u_1 \oplus u_2)W(y_2|u_2).$$

(c) (2 pts) Show that $Q(W^+) = Q(W)^2$.
Hint: $Q(W^+) = \sum_{y_1,y_2,u_1} \sqrt{W^+(y_1y_2u_1|0)W^+(y_1y_2u_1|1)}$

(d) (2 pts) Use the inequality

$$\sqrt{(ab+cd)(ac+bd)} \leq \left(\sqrt{ab} + \sqrt{cd}\right)\left(\sqrt{ac} + \sqrt{bd}\right) - 2\sqrt{abcd}$$

to show that $Q(W^-) \leq 2Q(W) - Q(W)^2$.

(e) (3 pts) Given a binary input channel W, Let $\tilde{W} = \text{BEC}(p)$, where $p = Q(W)$. Show that for any sign sequence $s^t \in \{+,-\}^t$, $Q(W^{s^t}) \leq Q(\tilde{W}^{s^t})$.