Quantum Field Theory

Homework Set 2

Exercise 1: Spinors

Consider a Weyl spinor transforming in the representation $(1/2, 0)$ of the Lorentz group ψ_L:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$
$$\psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1} x'),$$

where Λ_L is the Lorentz transformation in the representation $(1/2, 0)$. We have omitted the spinor index α for shortness.
Identify the representation of the Lorentz group the following term belongs to:

$$\psi_L^{\dagger} \bar{\sigma}^{\mu} \psi_L.$$

Consider now a pair of *left* Weyl spinors: ψ_L^1, ψ_L^2 and assume they form a dublet Ψ_L of an additional *Isospin* $SU(2)$ symmetry (don't confuse this $SU(2)$ with the Lorentz Group, it's an internal symmetry). In addition take a single *right* Weyl spinor ψ_R which is a singlet of the $SU(2)$ Isospin (ψ_R, Ψ_L are completely unrelated fields, they are not obtained one from the other using ε). Note that the doublet $\Psi_L = \binom{\psi_L^1}{\psi_L^2}$ has two indices

$$(\Psi_L)^a_{\alpha} \qquad \begin{array}{c} \alpha = 1, 2 \text{ represents the Lorentz index} \\ a = 1, 2 \text{ represents the Isospin index} \end{array}$$

while ψ_R has only the Lorentz $\dot{\beta}$ index, and each transformation acts separately on the two indices:

$$\text{Lorentz} \begin{cases} x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \\ (\Psi'_L)^a_{\alpha}(x') = (\Lambda_L)_{\alpha}{}^{\beta} (\Psi_L)^a_{\beta}(\Lambda^{-1} x') \\ (\psi'_R)^{\dot{\beta}}(x') = (\Lambda_R)^{\dot{\beta}}{}_{\dot{\delta}} (\psi_R)^{\dot{\delta}}(\Lambda^{-1} x'), \end{cases}$$

$$\text{Isospin} \begin{cases} x'^{\mu} = x^{\mu} \\ (\Psi'_L)^a_{\alpha}(x') = U^a_b (\Psi_L)^b_{\alpha}(x) \\ (\psi'_R)^{\dot{\beta}}(x') = (\psi_R)^{\dot{\beta}}(x). \end{cases}$$

- Identify the representation of Isospin and Lorentz group the following terms belong to:

  $$\psi_R^{\dagger} \Psi_L \,, \qquad [(\Psi_L)^a]^{\dagger} (\sigma^i)^a{}_b \, \bar{\partial} \Psi_L^b,$$

  where we have suppressed the Lorentz indices and here $\bar{\partial} \equiv \bar{\sigma}_{\mu} \partial^{\mu}$. Note that in the second term the Pauli matrix is contracted with the Isospin indices while $\bar{\partial}$ with the Lorentz ones.

Exercise 2: Yukawa coupling of the SM

Consider the following set of fields:

- an Isospin doublet of left spinors Ψ_L with U(1)-charge $+1/6$,
- an Isospin singlet right spinor u_R with U(1)-charge $+2/3$,
- an Isospin singlet right spinor d_R with U(1)-charge -1/3,
- an Isospin doublet of scalars Φ with U(1)-charge $+1/2$,

which have transformation properties under Lorentz, Isospin and an additional $U(1)$ summarized in the table:

Field	Lorentz	SU(2)	U(1)
Ψ_L	$\Lambda_L \Psi_L$	$U\Psi_L$	$e^{i\frac{1}{6}\alpha}\Psi_L$
u_R	$\Lambda_R u_R$	u_R	$e^{i\frac{2}{3}\alpha}u_R$
d_R	$\Lambda_R d_R$	d_R	$e^{-i\frac{1}{3}\alpha}d_R$
Φ	Φ	$U\Phi$	$e^{i\frac{1}{2}\alpha}\Phi$

- Show that, once we require the Lagrangian to be invariant under all the three symmetries and to contain terms with dimension less than or equal to four, the allowed interactions between scalars and fermions are restricted to only two terms (plus their hermitian conjugate). Find them. (We are interested only in terms containing *at least* one scalar and one fermion).