Quantum Field Theory

Set 3

Exercise 1: Higher derivative scalar theory

Consider the *Lagrangian density* of a real scalar field $\phi(\vec{x}, t)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

- Find the equation of motion of the field ϕ
- Specialize to the massive $\lambda\phi^4$-theory: $V = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$.

Add to the lagrangian density the term: $\alpha(\partial_\mu\phi\partial^\mu\phi)^2$

- Compute the dimension of the constant α.
- Find the equation of motion of the field ϕ.

Exercise 2: Hamiltonian formalism

Consider the action of the *massive $\lambda\phi^4$ scalar theory*:

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] .$$

- Find the equation of motion using the Hamiltonian formalism and show the equivalence with the Lagrangian formalism.

Exercise 3: Classical Electromagnetism

Consider the classical electromagnetic fields $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$.

- Write the Maxwell equations in presence of external an source.

Define the *field strength* $F_{\mu\nu}$ as the 4×4 matrix

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{bmatrix} \qquad F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}$$

- Recalling the definitions of the field in term of the vector potential $A_\mu = (A_0, A_i)$ show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Introduce the *Lagrangian density* of the electromagnetic field in the presence of an external source:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

  where J^μ is the current associated to the source. For a particle at rest $J^\mu = (e\delta^3(\vec{x} - \vec{x}_0), \vec{0})$.
  Find the Euler Lagrange equations and show that they correspond to the inhomogeneous Maxwell equations.

- Show that the additional two Maxwell equations follow from the *Bianchi identity*

$$\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} =?$$

- Find the static solution (independent of time) $A_\mu(\vec{x})$.