Quantum Field Theory

Set 4

Exercise 1: Classical field theory

Consider a scalar field ϕ with cubic potential interacting with an external source $J(t,\vec{x})$. The Lagrangian density has the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{6}\phi^3 - \phi J.$$

Find the equation of motion. Assuming that the parameter λ is small we can look for a solution of the equation of motion perturbatively. Substitute the ansatz $\phi = \phi_0 + \lambda\phi_1$ and factorize the terms with the same power of λ. Find the Green function of the problem (the solution of the equation $\Box\mathcal{G} = \delta(t)\delta^3(\vec{x})$) and use it to write the solution in the presence of a general source $J(x,t)$. Specialize to the simple source $J(x,t) = J\theta(t)\theta(\tau - t)\delta^3(\vec{x})$. (Hint: in order to solve the equation for the Green function go to Fourier space and write $\Box \to -((w+i\varepsilon)^2 - k^2)$. What's special in this particular choice of regularization?).
Consider the first order (in λ) equation and plug in the solution found for ϕ_0; then solve for ϕ_1.

Exercise 2: Affine transformations

Consider the two parameter Lie group acting on the real numbers defined by the transformations:

$$\mathcal{D}(a) : x \to ax \quad \mathcal{T}(b) : x \to x + b.$$

- Define
$$U(a,b) = \mathcal{T}(b)\mathcal{D}(a) : x \to x' = ax + b.$$
Show that this transformations defines a group.
- Consider the generators of the group T, D such that:
$$\mathcal{D}(1+a) = e^{iaD}, \qquad \mathcal{T}(b) = e^{ibT},$$
Find the commutation relations between the generators:
$$[T,T] =? \qquad [D,T] =? \qquad [D,D] =?$$

Exercise 3: Some Lie Groups

Consider the following Lie groups:

$$\begin{aligned}
&U(N) = \left\{N \times N \text{complex matrices such that } U^\dagger U = UU^\dagger = 1\right\},\\
&SU(N) = \left\{N \times N \text{complex matrices such that } U^\dagger U = UU^\dagger = 1,\, \det U = 1\right\},\\
&O(N) = \left\{N \times N \text{real matrices such that } R^T R = R^T R = 1\right\},\\
&SO(N) = \left\{N \times N \text{real matrices such that } R^T R = R^T R = 1,\, \det R = 1\right\},\\
&SL(N,\mathbb{C}) = \{N \times N \text{complex matrices such that } \det A = 1\ \}.
\end{aligned}$$

For each of them find a basis and the dimension of the Lie Algebra (recall that a Lie Algebra is a vector space).

Exercise 4: Jacobi Identity

Show that, for any Lie Algebra defined by the commutation relations

$$\left[T^a, T^b\right] = i f^{abc} T^c \ ,$$

the following *Jacobi identity* holds:

$$\sum_d \left(f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd}\right) = 0.$$

This is a straightforward consequence of the identity:

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = 0.$$