Quantum Field Theory

Set 19: solutions

Exercise 1

Given the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu + \bar{\psi}(i\partial\!\!\!/ - qA\!\!\!/)\psi,$$

the equations of motion for the vector field are:

$$-\partial_\mu(\partial^\mu A^\rho - \partial^\rho A^\mu) = M^2 A^\rho - J^\rho,$$

where $J^\rho \equiv q\bar{\psi}\gamma^\rho\psi$. In Fourier space they read:

$$[(k^2 - M^2)g^{\mu\rho} - k^\mu k^\rho]\tilde{A}_\mu(k) = -\tilde{J}^\rho(k).$$

Expanding for $k \ll M$, we get:

$$\tilde{A}_\mu(k) \simeq \frac{1}{M^2}\tilde{J}_\mu(k) \quad \Longrightarrow \quad A_\mu(x) \simeq \frac{1}{M^2}J_\mu(x) = \frac{q}{M^2}\bar{\psi}(x)\gamma_\mu\psi(x).$$

Note that the same result can be obtained by solving the equation of motion for the field A_μ without any approximation, and then taking the low energy limit of the solution. In this case we consider the Green's function $G_{\sigma\alpha}(x)$, satisfying the defining equation:

$$[-(\partial_\mu\partial^\mu + M^2)g^{\rho\sigma} + \partial^\rho\partial^\sigma]G_{\sigma\alpha}(x) = \delta^\rho_\alpha\delta^{(4)}(x).$$

To find the explicit form of the Green's function it is convenient to work in Fourier space, where the equation becomes $[(k^2 - M^2)g^{\rho\sigma} - k^\rho k^\sigma]\tilde{G}_{\sigma\alpha}(k) = \delta^\rho_\alpha$. Looking for a solution of the form $\tilde{G}_{\sigma\alpha}(k) = Ak_\sigma k_\alpha + Bg_{\sigma\alpha}$ (the only two tensor structures available), we get in the end:

$$\tilde{G}_{\sigma\alpha}(k) = \frac{1}{k^2 - M^2}\left(g_{\sigma\alpha} - \frac{k_\sigma k_\alpha}{M^2}\right).$$

The solution for the field A_μ is then given by the convolution of $G_{\sigma\alpha}(x)$ with J^α:

$$A_\mu(x) = -\int d^4y \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M^2}\left(g_{\mu\alpha} - \frac{k_\mu k_\alpha}{M^2}\right)e^{-ik(x-y)}J^\alpha(y).$$

In the low energy limit $k \ll M$ we obtain:

$$A_\mu(x) \simeq \int d^4y \int \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\alpha}}{M^2}e^{-ik(x-y)}J^\alpha(y) = \frac{J_\mu(x)}{M^2}.$$

Plugging this result in the equation of motion for the field ψ, namely $(i\partial\!\!\!/ - qA\!\!\!/)\psi = 0$, we find:

$$\left(i\partial^\mu - \frac{q^2}{M^2}\bar{\psi}\gamma^\mu\psi\right)\gamma_\mu\psi = 0,$$

which can be interpreted as derived from a *Fermi* effective Lagrangian:

$$\mathcal{L}_F = \bar{\psi}i\partial\!\!\!/\psi - \frac{q^2}{2M^2}\bar{\psi}\gamma^\mu\psi\ \bar{\psi}\gamma_\mu\psi.$$

Exercise 2

In general, a state with n-particles and m-antiparticles can be expressed as the superposition of eigenstates of the momentum:

$$|\Phi\rangle = \int d\Omega_{\vec{p}_1}...d\Omega_{\vec{p}_n}\, d\Omega_{\vec{q}_1}...d\Omega_{\vec{q}_m} f(\vec{p}_1,...,\vec{p}_n,\vec{q}_1,...,\vec{q}_m) a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_n)\, b^\dagger(\vec{q}_1)...b^\dagger(\vec{q}_m)|0\rangle.$$

In the simple case of a system consisting of a particle and an anti-particle in the center of mass ($\vec{p}_1 = -\vec{q}_1$) with a defined angular momentum l we have:

$$|\Phi_l\rangle = \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p}) a^\dagger(\vec{p}) b^\dagger(-\vec{p})|0\rangle,$$

where $f_l(\vec{p},-\vec{p})$ is the wave function describing a state with a given angular momentum (it is actually a superposition of spherical harmonics with total angular momentum l) and satisfies the property:

$$f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p})\ .$$

Let us now perform a parity transformation: in general each particle acquires a multiplicative phase η_P but since the antiparticle gets the same factor η_P and $\eta_P^2 = 1$ this factor never appears. In addition to this, the spatial momenta are inverted:

$$\begin{aligned} P|\Phi_l\rangle &= \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, P a^\dagger(\vec{p}) P^\dagger\, P b^\dagger(-\vec{p}) P^\dagger |0\rangle \\ &= \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, a^\dagger(-\vec{p})\, b^\dagger(\vec{p})|0\rangle \\ &= \int d\Omega_{\vec{p}}\ f_l(-\vec{p},\vec{p})\, a^\dagger(\vec{p})\, b^\dagger(-\vec{p})|0\rangle = (-1)^l|\Phi_l\rangle, \end{aligned}$$

where in the first line we have inserted $P^\dagger P = 1$ and we have used the invariance of the vacuum $P\,|0\rangle = |0\rangle$. Note also that $P^\dagger = P$, since we require that acting twice with parity has to be equal to the identity transformation, thus $POP^\dagger = P^\dagger OP$ for any operator O. Therefore a state made of a scalar particle-antiparticle pair with a given angular momentum changes by a factor $(-1)^l$ under parity.
Let's now consider a state consisting of a fermionic particle-antiparticle pair. We can write such a state as:

$$|\Psi_{l,S}\rangle = \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p}) \chi_S(r,t) \tilde{d}^\dagger(\vec{p},r) b^\dagger(-\vec{p},t)|0\rangle,$$

where the two functions satisfy:

$$f_l(\vec{p},-\vec{p}) = (-1)^l f_l(-\vec{p},\vec{p})\,, \qquad \chi_S(t,r) = (-1)^{S+1}\chi_S(r,t).$$

Notice that the transformation property for the spin function $\chi_S(r,t)$ reflects the fact that the product of two spin 1/2 states is symmetric if the total spin is 1 and is antisymmetric if the total spin is 0. Again we can apply the parity operator:

$$\begin{aligned} P|\Psi_{l,S}\rangle &= \sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, \chi_S(r,t)\, P\tilde{d}^\dagger(\vec{p},r) P^\dagger\, P b^\dagger(-\vec{p},t) P^\dagger|0\rangle \\ &= -\sum_{r,t} \int d\Omega_{\vec{p}}\ f_l(\vec{p},-\vec{p})\, \chi_S(r,t)\, \tilde{d}^\dagger(-\vec{p},r)\, b^\dagger(\vec{p},t)\,|0\rangle = (-1)^{l+1}|\Psi_{l,S}\rangle. \end{aligned}$$

Notice that P doesn't touch the spins.

Exercise 3

The transformation properties of a Weyl fermion under Charge-conjugation are:

$$\begin{aligned} C^\dagger\, \chi_L\, C &= \eta_L \epsilon \chi_R^*, \\ C^\dagger\, \chi_R\, C &= \eta_R \epsilon \chi_L^*. \end{aligned}$$

Let's apply them to the Lagrangian of a Dirac fermion:

$$
\begin{aligned}
C^\dagger \mathcal{L}\, C &= iC^\dagger\, \chi_L^\dagger\, C\,\bar\sigma^\mu\partial_\mu\, C^\dagger\,\chi_L\, C + iC^\dagger\,\chi_R^\dagger\, C\sigma^\mu\partial_\mu C^\dagger\,\chi_R\, C - m(C^\dagger\,\chi_R^\dagger\, C\, C^\dagger\,\chi_L\, C + h.c.)\\
&= i(\epsilon\chi_R^*)^\dagger\bar\sigma^\mu\partial_\mu\,\epsilon\chi_R^* + i(\epsilon\chi_L^*)^\dagger\sigma^\mu\partial_\mu\,\epsilon\chi_L^* - m(\eta_R^*\eta_L(\epsilon\chi_L^*)^\dagger\epsilon\chi_R^* + h.c)\\
&= i\chi_R^T\epsilon^T\bar\sigma^\mu\partial_\mu\,\epsilon\chi_R^* + i\chi_L^T\epsilon^T\sigma^\mu\partial_\mu\,\epsilon\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\epsilon^T\,\epsilon\chi_R^* + h.c)\\
&= i\chi_R^T(\sigma^\mu\partial_\mu)^T\,\chi_R^* + i\chi_L^T(\bar\sigma^\mu\partial_\mu)^T\,\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\chi_R^* + h.c).
\end{aligned}
$$

Where we have used $\epsilon^T(\bar\sigma^\mu)\epsilon = (\mathbb{1}_2, -\epsilon(\bar\sigma^i)\epsilon) = (\sigma^\mu)^T$ and $\epsilon^T\epsilon = \mathbb{1}_2$. At this point we can integrate the Lagrangian by parts (recall that it is the action that must be invariant under a symmetry):

$$
C^\dagger\mathcal{L}\, C = -i\partial_\mu\chi_R^T(\sigma^\mu)^T\,\chi_R^* - i\partial_\mu\chi_L^T(\bar\sigma^\mu)^T\,\chi_L^* - m(\eta_R^*\eta_L\chi_L^T\chi_R^* + h.c).
$$

In order to simplify we write the indices explicitly:

$$
\begin{aligned}
C^\dagger\mathcal{L}\, C &= -i\partial_\mu\chi_{R\alpha}(\sigma^\mu)^T_{\alpha\beta}\,\chi_{R\beta}^* - i\partial_\mu\chi_{L\alpha}(\bar\sigma^\mu)^T_{\alpha\beta}\,\chi_{L\beta}^* - m(\eta_R^*\eta_L\chi_{L\alpha}\chi_{R\alpha}^* + h.c)\\
&= -i\partial_\mu\chi_{R\alpha}(\sigma^\mu)_{\beta\alpha}\,\chi_{R\beta}^* - i\partial_\mu\chi_{L\alpha}(\bar\sigma^\mu)_{\beta\alpha}\,\chi_{L\beta}^* - m(\eta_R^*\eta_L\chi_{L\alpha}\chi_{R\alpha}^* + h.c)\\
&= i\chi_{R\beta}^*(\sigma^\mu)_{\beta\alpha}\,\partial_\mu\chi_{R\alpha} + i\chi_{L\beta}^*(\bar\sigma^\mu)_{\beta\alpha}\,\partial_\mu\chi_{L\alpha} + m(\eta_R^*\eta_L\chi_{R\alpha}^*\chi_{L\alpha} + h.c),
\end{aligned}
$$

where in the last step we have switched the order of the fermions and used the fact that two fermions anti-commute. Finally (up to total derivatives):

$$
C^\dagger\mathcal{L}\, C = i\chi_L^\dagger\,\bar\sigma^\mu\partial_\mu\,\chi_L + i\chi_R^\dagger\sigma^\mu\partial_\mu\chi_R + m(\eta_R^*\eta_L\chi_R^\dagger\chi_L + \eta_R\eta_L^*\chi_L^\dagger\chi_R).
$$

We see that the only way to achieve the invariance of the Dirac action is to impose $\eta_R^*\eta_L = -1$.
Note that this condition can also be easily obtained by noting that, applying twice the charge conjugation operator on a Weyl spinor, one should get back the spinor itself: $C^\dagger C^\dagger\chi_L CC = C^\dagger\eta_L\epsilon\chi_R^* C = \eta_L\eta_R^*\epsilon^2\chi_L = \chi_L$, which implies $\eta_R^*\eta_L = -1$ since $\epsilon^2 = -1$. Note also that, in order to satisfy the physical requirement $C^2 = 1$, it must be $C = C^\dagger$ (since C is unitary), as it is for parity.
On a Dirac spinor, the action of charge conjugation is

$$
C^\dagger\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}C = \underbrace{\begin{pmatrix}-\eta_L & 0\\ 0 & \eta_R\end{pmatrix}}_{\eta_C}\underbrace{i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}}_{\gamma^0\gamma^2}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_L^*\\ \chi_R^*\end{pmatrix}}_{\bar\psi^T},
$$

where we have used $\sigma^2 = -i\epsilon$. This proves that $U_C = i\gamma^0\gamma^2$. Note that the choice $\eta_L = -1$, $\eta_R = 1$, compatible with the constraint $\eta_R^*\eta_L = -1$, the matrix η_C can be eliminated from the formalism since it becomes the identity.

Exercise 4

The momentum $p^\mu = (E, 0, 0, p)$ and the polarization vector $\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0)$ satisfy the Lorentz-invariant constraint $p^\mu\varepsilon_\mu = 0$, in addition to the normalization conditions $\varepsilon^\mu\varepsilon_\mu = -1$ and $p^\mu p_\mu = M^2$.
ε^μ is an eigenvector of helicity with eigenvalue $+1$, as can be seen recalling the helicity operator:

$$
h \equiv \frac{\vec p\cdot\vec J}{|\vec p|} = J_3 = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},
$$

and by applying it on $\vec\epsilon \equiv (1, i, 0)$.
After a transverse boost in the y direction:

$$
\Lambda = \begin{pmatrix}\gamma & 0 & \gamma\beta & 0\\ 0 & 1 & 0 & 0\\ \gamma\beta & 0 & \gamma & 0\\ 0 & 0 & 0 & 1\end{pmatrix},
$$

we find:

$$
\begin{aligned}
p'^\mu &= (\gamma E, 0, \gamma\beta E, p)\,,\\
\varepsilon'^\mu &= \frac{1}{\sqrt{2}}\,(i\gamma\beta, 1, i\gamma, 0)\,.
\end{aligned}
$$

Note that, correctly, $p'^{\mu}\varepsilon'_{\mu} = 0$.
In order to decompose this vector on a basis of vectors with definite helicity, it is convenient to first rotate the three space in such a way as to align the new z direction to $\vec{p}'$, namely to perform the transformation:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{p}{\gamma k} & -\frac{\beta E}{k} \\ 0 & 0 & \frac{\beta E}{k} & \frac{p}{\gamma k} \end{pmatrix},$$

where $k \equiv \gamma^{-1}\sqrt{p^2 + (\gamma\beta E)^2}$. So we get:

$$\begin{aligned} \tilde{p}^{\mu} &= \gamma\,(E, 0, 0, k)\,, \\ \tilde{\varepsilon}^{\mu} &= \frac{1}{\sqrt{2}}\left(i\gamma\beta, 1, \frac{ip}{k}, \frac{i\gamma\beta E}{k}\right). \end{aligned}$$

The helicity basis is a set of polarization vectors $\tilde{\varepsilon}_{(i)}$ with definite helicity; they satisfy the transversality condition $\tilde{\varepsilon}^{\mu}_{(i)}\tilde{p}_{\mu} = 0, \;\; \forall\; i = -, 0, +$. In this frame they are:

$$\begin{aligned} \tilde{\varepsilon}^{\mu}_{(+)} &= \frac{1}{\sqrt{2}}\,(0, 1, i, 0)\,, \\ \tilde{\varepsilon}^{\mu}_{(-)} &= \frac{1}{\sqrt{2}}\,(0, 1, -i, 0)\,, \\ \tilde{\varepsilon}^{\mu}_{(0)} &= \frac{\gamma}{M}\,(k, 0, 0, E)\,, \end{aligned}$$

where the subscripts indicate the helicity eigenvalues.
Decomposing $\tilde{\varepsilon}'^{\mu}$ on this basis yields:

$$\tilde{\varepsilon}^{\mu} = \left(\frac{1 + p/k}{2}\right)\tilde{\varepsilon}^{\mu}_{(+)} + \left(\frac{1 - p/k}{2}\right)\tilde{\varepsilon}^{\mu}_{(-)} + \left(\frac{i\beta M}{\sqrt{2}k}\right)\tilde{\varepsilon}^{\mu}_{(0)}.$$

Note in particular that starting from a massive vector with positive helicity and performing a transverse boost, results in a superposition of all possible helicity states. This is different from the case of a massless vector. Indeed, it has been proven in Set17 (and it can be deduced here as well by taking the limit $M \to 0$) that for the massless case, starting with a positive helicity state, we end up with a positive helicity state (plus a longitudinal component).