Quantum Field Theory

Set 5: solutions

Exercise 1

The solution of this exercise in contained in the Notes on Lie Groups on the website.

Exercise 2

The explicit form of the three matrices is:

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad T^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \qquad T^3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The group $SO(3)$ is defined as

$$SO(3) = \left\{ R \in GL(3, \mathbb{R}) | \; RR^T = R^T R = 1, \; \det(R) = 1 \right\}$$

Parametrizing a general element of the group using the exponential function, $R(\alpha) = e^{i\alpha^a T^a}$, one can translate the constraints on the elements of the group to constraints on the generators:

$$1 = RR^T = (1 + i\alpha^a T^a)(1 + i\alpha^b (T^b)^T) + O(\alpha^2) \implies T^a = -(T^a)^T.$$

The Algebra of $SO(3)$ is a vector space generated by $3(3-1)/2 = 3$ antisymmetric objects, together with the usual commutator $[,]$. The three matrices defined at the beginning are

- antisymmetric,
- independent,
- in number equal to the dimension of the space.

Therefore they form a basis for (a representation of) the algebra $so(3)$. Having an explicit representation of the generators of a Lie Algebra, one can compute the commutators between them and extract the structure constants. The commutation relations which one obtains in this way are the same as in all the other representations, since the structure of the algebra of course doesn't depend on its explicit representation.
In the present case one has

$$\begin{aligned} \left[T^1, T^2\right] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = iT^3. \end{aligned}$$

Similarly one can explicitly compute

$$\left[T^2, T^3\right] = iT^1, \qquad \left[T^1, T^3\right] = -iT^2,$$

and identify the structure constant of the group $f^{abc} = \epsilon_{abc}$. This is the Algebra of the angular momentum one is used to deal with for example in quantum mechanics. The statement that a state $|s\rangle$ has angular momentum

J means that it belongs to a vector space on which acts a representation of the rotation group $SO(3)$ (call this representation j – we will see that representations can be labelled by an integer number). Under the action of the group, $|s\rangle$ transforms according to $|s\rangle \to e^{i\alpha_a T_a^{(j)}}|s\rangle$, where $T_a^{(j)}$ are the generators of $SO(3)$ in the representation j.
Coming back to structure constants, it is also possible to extract the commutation relations using the implicit form $(T^a)_i^j = -i\epsilon_{aij}$:

$$\begin{aligned}\left[T^a, T^b\right]_i^k &= (T^a)_i^j (T^b)_j^k - (T^b)_i^j (T^a)_j^k = (-i)^2 \epsilon_{aij}\epsilon_{bjk} - (-i)^2 \epsilon_{bij}\epsilon_{ajk} \\ &= \epsilon_{abc}\epsilon_{cik} = i\epsilon_{abc}(T^c)_i^k,\end{aligned}$$

where the last equality is a consequence of the identity $\epsilon_{aij}\epsilon_{bjk} + \epsilon_{ajk}\epsilon_{bji} + \epsilon_{abj}\epsilon_{jik} = 0$ (which in the end is the Jacobi identity for the structure constants of $so(3)$).
One can show that a general element of the group $SO(3)$ is a rotation acting on three dimensional vectors. To see this one can consider the *fundamental (or defining) representation*, that is to say the explicit representation of the group $SO(3)$ on $\mathbb{R}^3$ that we have previously recalled. An element of the group depends on three parameters α^a: one can collect them in a vector and call $\vec{n} = \vec{\alpha}/|\vec{\alpha}|$ the direction of this vector and $\theta = |\vec{\alpha}|$ the modulus of the vector. It's easy to prove that the action of the element $R(\alpha) = e^{i\alpha^a T^a}$ on a vector $\vec{x}$ corresponds to a rotation of this vector of an angle θ around the direction $\vec{n}$. One can firstly consider an infinitesimal rotation ($\theta << 1$)

$$\begin{aligned}&R(\alpha)_i^j\, x_j \simeq \left(1 + i\theta n^a T^a + O(\alpha^2)\right)_i^j\, x_j \simeq \left(\delta_i^j + i\theta n^a (T^a)_i^j +\right)\, x_j = x_i + \theta\epsilon_{aij} n^a x_j \\ &\Longrightarrow R(\alpha): \;\; \vec{x} \longrightarrow \vec{x} + \theta\, \vec{x} \wedge \vec{n}.\end{aligned}$$

One can verify that this is in accord with the usual way of representing a rotation: for example a rotation around the 3^{rd} direction by an angle θ produces a change in the $1,2$ plane according to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ x_2\cos\theta - x_1\sin\theta \\ x_3 \end{pmatrix} \simeq \begin{pmatrix} x_1 + x_2\theta \\ x_2 - x_1\theta \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where we have expanded the trigonometric functions for small angles.
One can do more: exponentiating the generators one can obtain the explicit form of an element of $SO(3)$ and compare it with a generic finite rotation. It's particularly easy to perform this computation in the simple case where the rotation is around one of the axes: let's take again the 3^{rd} direction for concreteness. Recognizing that

$$(T^3)^{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv A,$$

then

$$\begin{aligned}&R(\theta\vec{n}^3) = e^{i\theta T^3} = 1 + i\theta T^3 - \frac{1}{2}\theta^2 (T^3)^2 + \dots \\ &= iT^3\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right) + A\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + 1 - A \\ &= \begin{bmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

One immediately recognizes the usual form of a rotation by an angle θ in the $1-2$ plane.

Note. The group $SO(n)$, as well as other groups of linear transformations, is usually not defined in abstract by characterizing its elements g, but specifying the properties of one particular representation (the $fundamental$ or $defining$ representation): in the case of $SO(3)$ the fundamental representation contains the 3×3 orthogonal matrices with determinant $= 1$. This does not mean of course that the group has only that representation. For example, a quantity which is invariant under rotations transforms according to a one dimensional representation of $SO(3)$ in which the generators are identically $= 0$, while an object with angular momentum $j = 2$ transforms according to a five dimensional representation, i.e. a representation in which the transformations are represented by 5×5 matrices.

The rest of the exercise deals with another group, $SU(2)$, and the relation between this group and the group of rotations that we have analyzed in the first part. To begin with, one can recall the definition of the group as

$$SU(2) = \left\{ U \in GL(2, \mathbb{C}) |\ UU^+ = U^+U = 1,\ \det(U) = 1 \right\}.$$

Then one can consider the representation of the group acting on the vector space V defined to be:

$$V = \left\{ M \in M(2, \mathbb{C}) |\ M = M^+,\ \mathrm{Tr}(M) = 0 \right\},$$

that is to say the set of hermitian traceless matrices. One can verify that this vector space coincides with the one that defines the Lie Algebra of $SU(2)$. Indeed for infinitesimal transformations

$$\begin{aligned} &1 = U^\dagger U = (1 - i\alpha^a (T^a)^\dagger)(1 + i\alpha^b T^b) + O(\alpha^2) \implies T^a = (T^a)^\dagger, \\ &1 = \det(e^{i\alpha T}) = e^{i\alpha \mathrm{Tr}(T)} \implies \mathrm{Tr}(T) = 0, \end{aligned}$$

therefore the two vector spaces coincide. If one is able to find a basis of V this will also be a basis of the Lie Algebra of $SU(2)$. A basis of the vector space V is given for example by the three Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Having a basis of the Lie Algebra it's possible to compute the commutation relations as we did for $SO(3)$:

$$\begin{aligned} &\left[\sigma^1, \sigma^2\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma^3, \\ &\left[\sigma^2, \sigma^3\right] = 2i\sigma^1, \qquad \left[\sigma^1, \sigma^3\right] = -2i\sigma^2, \end{aligned}$$

therefore the matrices $\tau^a \equiv \sigma^a/2$ satisfy the algebra of $SU(2)$:

$$\left[\tau^a, \tau^b\right] = i\epsilon_{abc}\tau^c,$$

which is exactly the same of that one of $SO(3)$. This is something that happens frequently: given a Lie Group one and only one Lie Algebra is associated to it, however the converse in not true; given a Lie Algebra there exists unique a connected and simply connected Lie group associated to it, but there may exist other different groups without these constraints associated to the same algebra.

To summarize, we are considering a representation of a Lie Group on its Lie Algebra; this particular representation is called *adjoint representation*. The action of an element U of the group on an element M of the space V is as follows:

$$U:\ M \longrightarrow M' = UMU^\dagger$$

The above action defines a good representation since

- It's a linear application from V to V; indeed $(M')^\dagger = M'$ and $\mathrm{Tr}(M') = \mathrm{Tr}(UMU^\dagger) = \mathrm{Tr}(M) = 0$.
- It respect the composition of the group transformations:

$$\begin{aligned} &U_1:\ M \longrightarrow M' = U_1 M U_1^\dagger, \qquad U_2:\ M' \longrightarrow M'' = U_2 M' U_2^\dagger, \\ &U_2 \circ U_1 : M \longrightarrow (U_2 \circ U_1) M (U_2 \circ U_1)^\dagger = U_2 U_1 M U_1^\dagger U_2^\dagger = M''. \end{aligned}$$

Any hermitian traceless matrix can be written as a linear combination of elements of the basis:

$$M = \begin{bmatrix} y_3 & y_1 - iy_2 \\ y_1 + iy_2 & -y_3 \end{bmatrix} = y_i \sigma^i.$$

From the above equality one can argue that an element M can be associated to a thee-dimensional vector $\vec{y} = (y_1, y_2, y_3)$, which is the set of coordinates of the element M in the chosen basis. We know that a representation of a group is defined as a mapping between the group and the matrices acting on a vector space. After having chosen a basis one can also build the explicit form of the matrices associates to the element U of $SU(2)$. Here there is a scheme of the relations:

$$\begin{aligned}
\Psi &: \text{Group} \longrightarrow \text{Matrices acting on V} \\
&: U \longrightarrow R_i^j \\
U &: V \longrightarrow V \\
&: M = y_i\sigma^i \longrightarrow UMU^\dagger = \tilde{y}_i\sigma^i \\
R &: V \longrightarrow V \\
&: y_i \longrightarrow \tilde{y}_i = R_i^j y_j.
\end{aligned}$$

In order to get the form of the matrix R associated to a given element U one can consider an infinitesimal element of $SU(2)$ acting on M:

$$\begin{aligned}
UMU^\dagger &\simeq (1 + i\alpha^a\tau^a)y_i\sigma^i(1 - i\alpha^b\tau^b) = y_i\sigma^i + \frac{i}{2}\left[\sigma^a, \sigma^i\right]\alpha^a y^i + O(\alpha^2) \\
&= y_i\sigma^i + i(i)\epsilon_{aic}\sigma^c\alpha^a y_i = (y^c - \epsilon_{cai}\alpha^a y_i)\sigma^c = \tilde{y}^c\sigma^c.
\end{aligned}$$

Therefore the matrix R_i^j associated to the element of the group U is a rotation of an angle $\theta = |\vec{\alpha}|$ around the direction identified by $\vec{\alpha}$. One has to notice an important feature of this relation: the element of the group U and $-U$ induce the same changing for the vector $\vec{y}$, therefore they have the same representative. The representation map is not injective, even if it's surjective.

To summarize, we have shown that the group $SU(2)$ and $SO(3)$ have the same Lie Algebra, even if they are different groups. This implies that given a representation of the Algebra one has for sure a representation of $SU(2)$ (because is connected and simply connected) but not necessarily a representation of the group $SO(3)$. It may happen however that some vector space support both the representations, as we have seen. In particular the adjoint representation of $SU(2)$ (the one on it's Lie Algebra that we have considered in this exercise) provides automatically a representation of $SO(3)$.

Exercise 3

We now show how one can build an irreducible representation of the Algebra of $SU(2)$ and therefore also a representation of the Group. Given the commutation relations

$$\left[T^a, T^b\right] = i\epsilon_{abc}T^c,$$

one can compute the following

$$\begin{aligned}
\left[T^\pm, T^\pm\right] &= \frac{1}{2}\left[T^1 \pm iT^2, T^1 \pm iT^2\right] = \pm\frac{i}{2}\left[T^1, T^2\right] \pm \frac{i}{2}\left[T^2, T^1\right] = 0, \\
\left[T^+, T^-\right] &= \frac{1}{2}\left[T^1 + iT^2, T^1 - iT^2\right] = -\frac{i}{2}\left[T^1, T^2\right] + \frac{i}{2}\left[T^2, T^1\right] = T^3, \\
\left[T^3, T^\pm\right] &= \frac{1}{\sqrt{2}}\left[T^3, T^1 \pm iT^2\right] = \frac{1}{\sqrt{2}}\left[T^3, T^1\right] \pm \frac{i}{\sqrt{2}}\left[T^3, T^2\right] = \frac{iT^2 \pm T^1}{\sqrt{2}} = \pm T^\pm.
\end{aligned}$$

It's easy to show that the sum of squared generators commutes with all the generators

$$\begin{aligned}
\left[\sum_{a=1}^{3} T^aT^a, T^b\right] &= \sum_{a=1}^{3}\left(T^a\left[T^a, T^b\right] + \left[T^a, T^b\right]T^a\right) = i\epsilon_{abc}T^aT^c + i\epsilon_{abc}T^cT^a \\
&= i\epsilon_{abc}T^aT^c - i\epsilon_{cba}T^cT^a = 0.
\end{aligned}$$

The operator $J^2 = \sum_{a=1}^{3} T^aT^a$ commutes with all the generators of the Algebra, therefore commutes with the whole Group. In an irreducible representation Ψ one can use the Schur's Lemma to prove that J^2 has to be a multiple of the identity:

$$\Psi : T^a \longrightarrow \tau^a \qquad \Psi : J^2 \longrightarrow \sum_{a=1}^{3}\tau^a\tau^a = \mu^2 \times 1,$$

where μ is some constant that we will determine in the following.
Let us consider an irreducible representation where generators are represented by $\tau^{\pm}$, τ^3, $\tau^a\tau^a = \mu^2 \times 1$, and let us consider inside the vector space an eigenvector $|m\rangle$ of the generator τ^3 relative to the eigenvalue m:

$$\tau^3|m\rangle = m\,|m\rangle.$$

The action of one of the other generators $\tau^{\pm}$ sends $|m\rangle$ into another vector $|m'\rangle$ which one can show to be still an eigenvector of τ^3 but with a different eigenvalue:

$$\tau^3|m'\rangle = \tau^3\tau^{\pm}|m\rangle = \tau^{\pm}\tau^3|m\rangle + [\tau^3,\tau^{\pm}]|m\rangle = m\tau^{\pm}|m\rangle \pm \tau^{\pm}|m\rangle = (m\pm 1)\tau^{\pm}|m\rangle,$$

that is to say the $\tau^{\pm}$ generators acting on $|m\rangle$ change its eigenvalue by one unity. This is why they are called *raising and lowering operators*. More precisely, if we call $|m\pm 1\rangle$ the state normalized to one respect to a given scalar product, then

$$|\tau^{\pm}|m\rangle|^2 = \langle m|(\tau^{\pm})^{\dagger}\tau^{\pm}|m\rangle = \frac{1}{2}\langle m|(\tau^1)^2 + (\tau^2)^2 \pm i[\tau^1,\tau^2]|m\rangle = \frac{1}{2}\langle m|\mu^2 - (\tau^3)^2 \mp \tau^3|m\rangle = \frac{1}{2}(\mu^2 - m(m\pm 1))$$

where it has been used $(\tau^{\pm})^{\dagger} = \tau^{\mp}$. Therefore the correct normalization is

$$\tau^{\pm}|m\rangle = \frac{1}{\sqrt{2}}\sqrt{\mu^2 - m(m\pm 1)}|m\pm 1\rangle.$$

Moreover, from the previous equalities one can argue that $\mu^2 - m(m\pm 1) \geq 0$, since we deal with a space with positive definite norm ($|\tau^{\pm}|m\rangle|^2 \geq 0$). At the end

$$m^2 + |m| \leq \mu^2.$$

This statement has two important consequences: firstly it's a proof that μ^2 is a positive quantity, and secondly it imposes a limit on the dimension of an irreducible representation: indeed starting from a given state $|m-\rangle$ one can apply the raising operator to get another state, independent from the original one. This will increase also the value of m of one unity. If one were free to keep on applying τ^+ he would end with a violation of the inequality (note that since the Casimir operator $(\tau)^2$ is proportional to the identity, its eigenvalue μ^2 is constant, i.e. does not depend on m). Hence the action of the raising operator has to give a null state at a certain point. This happens only when $m(m+1) = m_{max}(m_{max}+1) = \mu^2$. Starting from the state $|m_{max}\rangle$ one can apply the lowering operator to decrease the value of m. As before after a finite number of steps one has to find a null state

$$(\tau^-)^{n+1}|m_{max}\rangle \propto \tau^-|m_{max} - n\rangle = 0 \qquad \text{for some } n,$$

and this will happen when $(m-n)(m-n-1) = m_{min}(m_{min}-1) = \mu^2$. Matching the two relations one finds

$$m_{min}(m_{min}-1) = m_{max}(m_{max}+1) \implies m_{max} = -m_{min}.$$

Moreover m_{min} has been obtained starting from m_{max} with an integer number of steps equal to $2m_{max}+1$. This restricts the value of m_{max} to be a positive integer or semi-integer. Summarizing, using the notation $m_{max} = j$, an irreducible representation of the Algebra of $SU(2)$ is characterized by

- A vector space with dimension $2j+1$ with a basis given by the eigenvectors of τ^3:

$$\{|m\rangle\}\,, \qquad -j \leq m \leq j.$$

- The generators on this vector space are represented as follows

$$\begin{aligned}
&\tau^3|m\rangle = m|m\rangle,\\
&\sum_{a=1}^{3}\tau^a\tau^a|m\rangle = \mu^2|m\rangle = j(j+1)|m\rangle,\\
&\tau^{\pm}|m\rangle = \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m(m\pm 1)}|m\pm 1\rangle.
\end{aligned}$$

As already said, these are representation of the algebra and therefore also of the $SU(2)$ group. Not all of them are representations of $SO(3)$. The problem arises when one tries to pass from the algebra (which is somehow a local representation of the group) to a global representation of the group. $SO(3)$ has indeed the property that a rotation of 2π around any axis must coincide with the identity. This restricts the value of j to be only integer (we will see it explicitly in some example).

Finally one can consider some representation:

- $j = 0$ is the trivial representation and is called *scalar representation*.
- $j = 1/2$ is the first non trivial one. It's only a representation of $SU(2)$ and is called *spinorial representation*. It's composed by two states labelled by the value of j and m: $|j = 1/2, m = \pm 1/2\rangle$.
- $j = 1$ is a representation of both groups. It is called *vectorial representation* and corresponds to the adjoint of $SU(2)$ or the fundamental of $SO(3)$. A basis for this representation is given by three states labelled by
$$|1, 1\rangle,\ |1, 0\rangle,\ ,|1, -1\rangle.$$

More about $SU(2)$ and $SO(3)$

The Pauli matrices have many properties: in addition to the fact that they satisfy the algebra of $SU(2)$ we can easily show that they satisfy a different algebra, that involves the anticommutators of two matrices $\{A, B\} = AB + BA$. Indeed
$$\{\sigma^a, \sigma^b\} = 2\delta^{ab}.$$
as one can directly verify. The above relation is called *Clifford's Algebra*. Note that we are not claiming that any representation of the algebra of $SU(2)$ satisfy also the Clifford's one. This is only a peculiarity of Pauli matrices and therefore holds only when we consider the space of 2×2 hermitian traceless matrices, not general representations.
Using the commutator and anticommutator one can easily write the product of two Pauli matrices in terms of one:
$$\sigma^a\sigma^b = \frac{1}{2}\{\sigma^a\sigma^b\} + \frac{1}{2}[\sigma^a\sigma^b] = \delta^{ab} \times 1_2 + i\epsilon_{abc}\sigma^c.$$

The above expression allows one to exponentiate immediately an element of the $SU(2)$ algebra and get the explicit form of an element of the group:
$$\begin{aligned}
\frac{i^{2n}}{2^{2n}(2n)!}\alpha^{a_1}...\alpha^{a_{2n}}\,\sigma^{a_1}...\sigma^{a_{2n}} &= \frac{i^{2n}}{2^{2n}(2n)!}\alpha^{a_1}...\alpha^{a_{2n}}\,\sigma^{a_3}...\sigma^{a_{2n}}(\delta^{a_1a_2} \times 1_2 + i\epsilon_{a_1a_2c}\sigma^c) \\
&= \frac{i^{2n}|\vec{\alpha}|^2}{2^{2n}(2n)!}\alpha^{a_3}...\alpha^{a_{2n}}\,\sigma^{a_3}...\sigma^{a_{2n}} = \frac{i^{2n}|\vec{\alpha}|^{2n}}{2^{2n}(2n)!} \times 1_2, \\
\frac{i^{2n+1}}{2^{2n+1}(2n+1)!}\alpha^{a_1}...\alpha^{a_{2n+1}}\,\sigma^{a_1}...\sigma^{a_{2n+1}} &= \frac{i^{2n+1}|\vec{\alpha}|^{2n}}{2^{2n+1}(2n+1)!}\alpha^{a_{2n+1}}\sigma^{a_{2n+1}}.
\end{aligned}$$

Therefore an element of the group becomes
$$\begin{aligned}
U(\alpha) &= e^{i\alpha^a\sigma^a/2} = 1 + i\frac{\alpha^a}{2}\sigma^a - \frac{1}{8}\alpha^a\alpha^b\sigma^a\sigma^b + ... = 1_2 \times \left(1 - \frac{|\vec{\alpha}|^2}{4 \cdot 2!} + \dots\right) + i\sigma^a\frac{\alpha^a}{|\vec{\alpha}|} \cdot \left(\frac{|\vec{\alpha}|}{2} - \frac{|\vec{\alpha}|^3}{8 \cdot 3!}\dots\right) \\
&= \cos\left(\frac{|\vec{\alpha}|}{2}\right) \times 1_2 + in^a\sigma^a \sin\left(\frac{|\vec{\alpha}|}{2}\right) \equiv k_0 \times 1_2 + ik_i\sigma^i.
\end{aligned}$$

where n^a is the unitary vector pointing in the same direction as α^a. One can see that the general element of the group is a linear combination of the identity and of the Pauli matrices. The coefficients of the linear combination are not independent since they must respect the determinant constraint:
$$1 = \det\begin{bmatrix} k_0 + ik_3 & ik_1 + k_2 \\ ik_1 - k_2 & k_0 - ik_3 \end{bmatrix} = k_0^2 + k_1^2 + k_2^2 + k_3^2.$$

The above expression is the equation that defines the embedding of a 3-sphere into $\mathbb{R}^4$. This parametrization shows that the group $SU(2)$, thought of as a manifold, is equivalent to S^3, which is a connected simply connected

manifold.
Coming back to the first exercise one should recall that (the defining representation of) the group $SO(3)$ coincides with the adjoint representation of $SU(2)$. This representation is not injective because it associates two distinct elements of $SU(2)$ (U and $-U$) to the same element of $SO(3)$ (we say that $SU(2)$ is the double covering of $SO(3)$). This means that in order to visualize $SO(3)$ as a manifold one can think about a sphere where we identify a point with the opposite one: $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \sim -(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. The manifold obtained is usually denoted as $\frac{S^3}{\mathbb{Z}_2}$. This manifold is locally equivalent to the sphere, in particular they have the same tangent space, and this reflects the fact that the Algebras of $SO(3)$ and $SU(2)$ are the same. However the identification of opposite points has a crucial global consequence: this manifold is not simply connected (recall that a connected space is said simply connected if any closed curve can be continuously shrunk to a point). To see this, imagine a curve starting at the North Pole and ending at the South Pole. Since the starting and ending points are identified this curve is *close*. The considered curve however cannot be shrunk to a point without opening it, because as soon as we move one of the Poles the curve stops to be closed. To summarize the relation between the two groups is

$$SO(3) = \frac{SU(2)}{\mathbb{Z}_2}.$$

For compliteness we define the group $\mathbb{Z}_2$, which is the pair $\{-1, 1\}$ together with the usual multiplication.

Exercise 4

- Each element of the representation can be put in the following block triangular form

$$\begin{pmatrix} \mathbf{A}_{(N-m)\times(N-m)} & \mathbf{B}_{(N-m)\times m} \\ \mathbf{0}_{m\times(N-m)} & \mathbf{C}_{m\times m} \end{pmatrix}. \tag{1}$$

  Since the matrix is unitary its rows are an orthonormal basis for the vector space over which the representation act. In this notation the invariant subspace of the representation is the set of vectors which are zero in their last m rows. Notice also that the last m row-vectors defined by the matrix $\mathbf{C}$ are an orthonormal basis of the orthogonal complement of the invariant subspace of the representation. In particular the $N-m$ m-dimensional vectors define by $\mathbf{B}$ are orthogonal to each element of this basis, hence they vanish and $\mathbf{B} = 0$.

- Showing that the direct sum is a representation is trivial. Write

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2}$$

  For hypothesis the two matrices

$$AD = \begin{pmatrix} A_{11}D_1 & A_{12}D_2 \\ A_{21}D_1 & A_{22}D_2 \end{pmatrix}, \qquad DA = \begin{pmatrix} D_1A_{11} & D_1A_{12} \\ D_2A_{21} & D_2A_{22} \end{pmatrix}. \tag{3}$$

  are equal. Given that D_1 and D_2 are inequivalent, the equality of the off-diagonal elements $A_{12}D_2 = D_1A_{12}$ and $A_{21}D_1 = D_2A_{21}$ imply, by the second Shur's lemma, $A_{12} = A_{21} = 0$. Given that D_1 and D_2 are irreducible, the equality of the diagonal elements $A_{11}D_1 = D_1A_{11}$, $A_{22}D_2 = D_2A_{22}$ imply, by the first Shur's lemma, that $A_{11} = \lambda_1 I$ and $A_{22} = \lambda_2 I$.