INTEGRAL CONVEXITY AND PARABOLIC SYSTEMS

VERENA BÖGELEIN, BERNARD DACOROGNA, FRANK DUZAAR, PAOLO MARCELLINI, AND CHRISTOPH SCHEVEN

ABSTRACT. In this work we give optimal, i.e. necessary and sufficient, conditions for integrals of the Calculus of Variations to guarantee the existence of solutions – both *weak* and *variational solutions* – to the associated L^2-gradient flow. The initial values are merely L^2-functions with possibly infinite energy. In this context, the notion of *integral convexity*, i.e. the convexity of the variational integral and not of the integrand, plays the crucial role; surprisingly, this type of convexity is weaker than the convexity of the integrand. We demonstrate this by means of certain quasi-convex and non-convex integrands.

CONTENTS

1. INTRODUCTION

The aim of our research is twofold. First, we have in mind to introduce a model to handle evolutionary problems in *nonlinear elasticity*, following the celebrated approach introduced by Ball [4] in 1977 for the stationary case. Secondly, we refer to L.C. Evans, who stated in 2013 that one of the most fundamental open problems for *quasi-convex* variational integrals

$$\mathbf{F}(u) := \int_\Omega f(Du)\,\mathrm{d}x$$

is the study of *existence, uniqueness* and *regularity issues* for the L^2-gradient flow associated to $\mathbf{F}$

$$\partial_t u - \operatorname{div}\big(D_\xi f(Du)\big) = 0.$$

Date: September 13, 2019.

The precise references is [19, Section 1c, (ii)]; see also the introduction in [20] for a similar statement. In this paper we give a partial answer to these questions for general integral functionals of the Calculus of Variations of the form

$$\mathbf{F}(u) := \int_\Omega f(x, u, Du)\,\mathrm{d}x. \tag{1.1}$$

Here $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand, quasi-convex with respect to the gradient variable. We work in the context of *variational solutions*, which have been introduced by Lichnewsky & Temam [30]. The related theory is the natural parabolic counterpart to the classical Calculus of Variations in the stationary setting. Indeed, for parabolic problems variational solutions play the same role as minimizers for integral functionals. For the precise notion we refer to Definition 2.1 below. In this context of lack of convexity of the integrand we investigate necessary and sufficient conditions for the existence of variational solutions. One of the main results of this paper is:

Theorem 1.1. *Let $\mathbf{F}\colon W^{1,p}(\Omega, \mathbb{R}^N) \to [0, \infty]$ with $p > 1$ be a coercive integral functional as in (1.1) with a quasi-convex Carathéodory integrand $f(x, u, \xi)$. Then, there exists a variational solution to the L^2-gradient flow associated to $\mathbf{F}$ for any initial datum $u_o \in L^2(\Omega, \mathbb{R}^N)$ if and only if $\mathbf{F}$ is convex.*

Of course, to the list of assumptions on a quasi-convex integrand f certain growth conditions have to be added in order to guarantee the lower semi-continuity of $\mathbf{F}$; cf. [1, 33]. Precise statements will be given in § 2. At first glance, one might think that the convexity of the integral – in the sequel we denote this by the notion *integral convexity* – implies the convexity of the integrand f with respect to (u, Du). However, this is not the case. We emphasize, that in the vectorial setting $N > 1$ a quadratic integrand of the type

$$f(x, u, \xi) = \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a^{ij}_{\alpha\beta}(x)\xi^\alpha_i \xi^\beta_j + \sum_{\alpha,\beta=1}^{N} c_{\alpha\beta}(x)u^\alpha u^\beta, \tag{1.2}$$

whose coefficients $a^{ij}_{\alpha\beta}(x)$ satisfy the *Legendre-Hadamard condition*

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a^{ij}_{\alpha\beta}(x)\eta_i\eta_j\vartheta^\alpha\vartheta^\beta \geq \nu|\eta|^2|\vartheta|^2 \qquad \forall \eta \in \mathbb{R}^n, \vartheta \in \mathbb{R}^N$$

for some $\nu > 0$, leads to a convex integral $\mathbf{F}$; cf. Section 4.2. Indeed, if the coefficients $a^{ij}_{\alpha\beta}(x)$ are uniformly continuous and if the lower order coefficients $c_{\alpha\beta}(x)$ are chosen large enough in the sense of positive definite matrices it can be established that the variational integral $\mathbf{F}$ satisfies Gårding's inequality, which implies the integral convexity of $\mathbf{F}$. This is essentially the content of Theorem 4.6. Note that f in (1.2) is quasi-convex but not necessarily convex. Of course, this specific integrand can be perturbed by adding any non-negative Carathéodory integrand $g\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, \infty)$ which is convex with respect to u and Du. Other prototypes of non-convex integrands which are integral convex are

$$f(x, u, \xi) = |\xi|^2 + b(x)\big(|u|^2 - 1\big)^2 \quad \text{and} \quad f(x, u, \xi) = |\xi|^2 - c(x)|u|^2 \tag{1.3}$$

with $b(x) \geq 0$ and $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$ small enough. For the precise statement and more general cases we refer to § 5.

The above considerations show the importance of the notion of integral convexity in the context of evolutionary variational problems. The main assertion of Theorem 1.1 remains true also in the more general setting of functionals $\mathbf{F}\colon W^{1,p}(\Omega, \mathbb{R}^N) \to [0, \infty]$ with $p > 1$, which are coercive and sequentially lower semi-continuous with respect to weak convergence in $W^{1,p}(\Omega, \mathbb{R}^N)$. For such functionals we prove:

Theorem 1.2. *For any initial datum $u_o \in L^2(\Omega, \mathbb{R}^N)$ there exists a variational solution to the gradient flow associated to $\mathbf{F}$ if and only if $\mathbf{F}$ is convex.*

Some comments on the assumptions of Theorem 1.2 might be helpful. For a general Carathéodory integrand $f(x,u,\xi)$ the lower semi-continuity of the associated integral functional $\mathbf{F}$ with respect to the weak*-$W^{1,\infty}(\Omega,\mathbb{R}^N)$ topology implies the quasi-convexity of the integrand f. This is a well known result by Morrey in his pioneering work in 1952 [36]; see also [14, Second Edition; Theorem 8.1] and [24, Theorem 5.2]. On the contrary, under certain growth assumptions on the quasi-convex integrand the lower semi-continuity of the associated integral functional $\mathbf{F}$ can be deduced; cf. [1, 33].

The above arguments show that the quasi-convexity assumption in a certain sense is a necessary condition for the existence of variational solutions associated to energy integrals of the Calculus of Variations. However, Theorem 1.1 shows that quasi-convexity alone is not enough to guarantee the existence of variational solutions; in fact we establish in § 4.1 that quasi-convexity of the integrand in general does not imply integral convexity even with fixed boundary data. Therefore one has to impose integral convexity. This is the essential difference to the stationary setting in which quasi-convexity – and not integral convexity – is the main assumption for existence of minimizers.

The whole picture will be completed by the fact that certain standard growth assumptions on the integrand f allow to derive that variational solutions are indeed solutions of the associated parabolic system. This means that we can prove the existence of weak solutions to the L^2-gradient flow for some classes of quasi-convex integrals as in (1.1). That is, they weakly satisfy the parabolic system

$$\partial_t u - \operatorname{div}\big(D_\xi f(x,u,Du)\big) + D_u f(x,u,Du) = 0.$$

In summary, our results can be interpreted as the parabolic analogue of the existence theorem for quasi-convex integrals in the Calculus of Variations, in combination with the characterization of those integral functionals which are weakly lower semi-continuous. Indeed, quasi-convex integrands satisfying certain growth and coercivity assumptions are weakly lower semi-continuous and this allows to derive general existence theorems for minimizers by the *direct methods* of the Calculus of Variations. Vice versa, if the integral functional is weakly lower semi-continuous, then the integrand is quasi-convex and, under certain growth conditions, the minimizers solve the associated Euler-Lagrange system. On the contrary we cannot expect in general that weak solutions minimize the associated variational integral, unless the integrand is convex with respect to u and Du or, more generally, under *integral convexity*. Our paper shows that the same conclusion can be drawn for parabolic problems: if the integral is convex, then weak solutions of the parabolic Euler-Lagrange system are in fact variational solutions, see § 7.4. Finally, we emphasize that variational solutions are unique provided $\mathbf{F}$ is convex; this differs from the stationary case, where strict convexity of $\mathbf{F}$ is needed.

Let us briefly recall some prior work related to this research. We already mentioned the paper by Lichnewsky & Temam [30] related to evolutionary minimal surfaces. Necessary conditions for existence of parabolic minimizers have been derived by Wieser [43] in the case $p = 2$ under standard growth conditions; see also Daneri & Savaré [17]. Existence results, for instance in a metric context, have been obtained by Ambrosio [2], Ambrosio, Gigli & Savaré [3], Mielke & Stefanelli [35], and Rossi, Savaré, Segatti & Stefanelli [40]; for weak and variational solutions we refer to [8, 9, 11]. The innovation of this paper is the precise characterization of those not necessarily convex integrands from the Calculus of Variations for which existence of unique variational solutions holds. Moreover, we deal with $L^2(\Omega)$-initial data with possibly infinite energy.

In contrast, Müller, Rieger & Šverăk constructed in [39] a smooth quasi-convex integrand $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with quadratic growth and inhomogeneity $g\colon \Omega_T \to \mathbb{R}^2$, such that the associated Cauchy-Dirichlet problem with $u = 0$ on the parabolic boundary of Ω_T has at least two solutions. One of the solutions is nowhere of class C^1 in Ω_T, while the other one is smooth. Due to the non-uniqueness the associated integral functional can not be integral convex.

Finally, we mention that linear parabolic systems with principal part satisfying the strict Legendre–Hadamard condition have been treated with different approaches in the literature. For instance, in [25] Dong & Kim considered the case of second order linear parabolic systems with leading coefficients satisfying bounded mean oscillation (BMO) and vanishing mean oscillation (VMO) in the spatial variables, by means of methods introduced by Krylov [28, 29] for linear elliptic and parabolic systems in non-divergence form; see also [21, § 9] for similar results. We also quote [26] from the same authors; they deal with higher-order linear parabolic systems in the general context of the strict Legendre-Hadamard condition. Similar results for higher order systems are due to Boccia [5], Boccia & Krylov [6] and Gallarati & Veraar [22]. Finally, we refer to [12, 31] for classical existence results related to maximal monotone operators.

We conclude this introduction recalling a result by Evans [18] concerning partial regularity of minimizers of quasi-convex integrands: is it possible to prove a similar result in the context of parabolic variational solutions?

Acknowledgments. V. Bögelein has been supported by the FWF-Project P31956-N32 "Doubly nonlinear evolution equations". P. Marcellini is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probalità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

2. Notation and Results

Throughout the paper we denote by Ω a bounded domain in $\mathbb{R}^n$ with $n \in \mathbb{N}$. By $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ we denote the space-time cylinder with base Ω. We consider functionals $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ with $p>1$ and Dirichlet boundary values

$$u_* \in L^2(\Omega,\mathbb{R}^N) \cap W^{1,p}(\Omega,\mathbb{R}^N) \quad \text{with } \mathbf{F}(u_*) < \infty. \tag{2.1}$$

We assume that $\mathbf{F}$ satisfies the following conditions

$$\begin{cases} \mathbf{F}[u] \ge \nu \|Du\|^p_{L^p(\Omega,\mathbb{R}^N)} & \text{for all } u \in u_* + W^{1,p}_0(\Omega,\mathbb{R}^{N\times n}), \\ \mathbf{F} \text{ is sequentially lsc w.r.t. weak convergence on } u_* + W^{1,p}_0(\Omega,\mathbb{R}^{N\times n}), \\ \mathbf{F} \text{ is finite on } u_* + C^\infty_0(\Omega,\mathbb{R}^N). \end{cases} \tag{2.2}$$

In particular, $(2.2)_1$ implies that $\mathbf{F}$ is coercive in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$, i.e. that the level sets of $\mathbf{F}$ are bounded in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$.

Definition 2.1. Suppose that $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ is a variational functional, $u_* \in W^{1,p}(\Omega,\mathbb{R}^N)$ with $\mathbf{F}(u_*) < \infty$ and $u_o \in L^2(\Omega,\mathbb{R}^N)$. A measurable map $u\colon \Omega_T \to \mathbb{R}^N$ in the class

$$u \in C^0\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T; u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$$

is a *variational solution* associated to the functional $\mathbf{F}$ if and only if the variational inequality

$$\begin{aligned} \int_0^\tau \mathbf{F}(u)\,\mathrm{d}t \le \int_0^\tau \mathbf{F}(v)\,\mathrm{d}t + \iint_{\Omega_\tau} \partial_t v \cdot (v-u)\,\mathrm{d}x\mathrm{d}t \\ + \tfrac12 \|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac12 \|(v-u)(\tau)\|^2_{L^2(\Omega)} \end{aligned} \tag{2.3}$$

holds true, for any $\tau \in (0,T]$ and any $v \in L^p(0,T; u_* + W^{1,p}_0(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) \in L^2(\Omega,\mathbb{R}^N)$. □

Remark 2.2. Note that the variational inequality (2.3) implies that the initial values are assumed, in the sense that $u(0) = u_o$. This will be shown in § 7.2.6.

Now we are in the position to present the precise formulation of our main result.

Theorem 2.3. *Assume that* $p > 1$ *and that* u_* *and* $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ *satisfy hypothesis* (2.1) *and* (2.2)*. Then, for any* $u_o \in L^2(\Omega,\mathbb{R}^N)$ *there exists a unique variational solution* u *to the gradient flow for* $\mathbf{F}$ *in the sense of Definition* 2.1 *if and only if* $\mathbf{F}$ *is convex on* $L^2(\Omega,\mathbb{R}^N) \cap u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$*.*

Note that Theorems 1.1 and 1.2 are direct consequences of Theorem 2.3. The necessity part of the Theorem will be proven in § 6, while the sufficiency part will be shown in § 7. In § 7.1 we first consider the special case $u_o = u_*$. Here, we proceed by the so called method of elliptic regularization. The general case will be treated in § 7.2 with the method of minimizing movements.

Remark 2.4. The variational solution from Theorem 2.3 additionally satisfies $\partial_t u \in L^2(\Omega\times(\varepsilon,T],\mathbb{R}^N)$ and $Du \in L^\infty(\varepsilon,T;W^{1,p}(\Omega,\mathbb{R}^{N\times n}))$ for any $\varepsilon \in (0,T)$; see § 7.2.

Remark 2.5. In § 7.1 we will prove the sufficiency part of Theorem 2.3 under the stronger assumption $u_o = u_*$. In this case hypothesis $(2.2)_3$ is not necessary. Moreover, the variational solution u satisfies $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ and $Du \in L^\infty(0,T;W^{1,p}(\Omega,\mathbb{R}^{N\times n}))$; see also Lemma 3.4. In this case the variational inequality (2.3) we can be re-written as

$$\int_0^T \mathbf{F}(u)\,\mathrm{d}t \le \int_0^T \mathbf{F}(v)\,\mathrm{d}t + \iint_{\Omega_T} \partial_t u\cdot(v-u)\,\mathrm{d}x\mathrm{d}t$$

for any $v \in L^2(\Omega_T,\mathbb{R}^N) \cap L^p(0,T;u_* + W_0^{1,p}(\Omega,\mathbb{R}^N))$.

As one particular case we deduce in § 4 the existence of weak solutions u to Cauchy-Dirichlet problems associated to a class of parabolic linear systems

$$u_t - \operatorname{div}\big(A(x)D_x u\big) + C(x)u = h \quad \text{in } \Omega_T,$$

under the *Legendre-Hadamard* rank-one condition for $A(x) = \big(a_{ij}^{\alpha\beta}(x)\big)$ and the positivity of the matrix $C(x) = \big(c^{\alpha\beta}(x)\big)$. To perform this existence result we study the convexity of the quadratic energy-integral

$$\mathbf{F}(u) = \int_\Omega \big[\langle A(x)Du(x), Du(x)\rangle + \langle C(x)u, u\rangle\big]\,\mathrm{d}x.$$

We also consider some properties related to the convexity of the energy integral in some other non-quadratic cases; see § 4.1.

3. Preliminary lemmas

Before starting with the proof, we give two preliminary results which will be needed. The first result is a lower semi-continuity result for the integral functional $\mathbf{F}$ with respect to strong convergence in L^1. The precise result is

Lemma 3.1 (lower semi-continuity)**.** *Let* $p > 1$ *and* $u_* \in L^2(\Omega,\mathbb{R}^N)\cap W^{1,p}(\Omega,\mathbb{R}^N)$*, and assume that* $\mathbf{F} : W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ *is coercive in* $u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$ *and sequentially lower semi-continuous with respect to the weak topology in* $u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$*. Then, for any sequence* $\{u_k\}_{k\in\mathbb{N}} \subset L^p(0,T;u_* + W_0^{1,p}(\Omega,\mathbb{R}^N))$ *with* $u_k \to u$ *strongly in* $L^1(\Omega_T,\mathbb{R}^N)$*, we have*

$$\int_0^T \mathbf{F}(u)\,\mathrm{d}t \le \liminf_{k\to\infty} \int_0^T \mathbf{F}(u_k)\,\mathrm{d}t.$$

Proof. Without loss of generality we may assume that

$$\liminf_{k\to\infty} \int_0^T \mathbf{F}(u_k)\,\mathrm{d}t < \infty,$$

since otherwise there is nothing to prove. By Fatou's lemma we have

$$\int_0^T \liminf_{k\to\infty} \mathbf{F}(u_k)\,\mathrm{d}t \le \liminf_{k\to\infty} \int_0^T \mathbf{F}(u_k)\,\mathrm{d}t < \infty, \tag{3.1}$$

so that $\liminf_{k\to\infty} \mathbf{F}(u_k(t)) < \infty$ for a.e. $t \in [0,T]$. This means that for a.e. $t \in [0,T]$ there exists a subsequence $\mathcal{K}(t) \subset \mathbb{N}$ such that

$$\lim_{\mathcal{K}(t)\ni k\to\infty} \mathbf{F}(u_k(t)) = \liminf_{k\to\infty} \mathbf{F}(u_k(t)) < \infty.$$

The coercivity of $\mathbf{F}$ and Poincaré's inequality then ensure that

$$\lim_{\mathcal{K}(t)\ni k\to\infty} \|u_k(t)\|_{W^{1,p}(\Omega,\mathbb{R}^N)} < \infty.$$

Therefore, there exists another subsequence $\mathcal{K}_1(t) \subset \mathcal{K}(t)$ such that

$$u_k(t) \rightharpoonup u(t) \quad \text{weakly in } W^{1,p}(\Omega,\mathbb{R}^N) \text{ as } \mathcal{K}_1(t) \ni k \to \infty.$$

Note that the limit $u(t)$ is uniquely determined due to the strong convergence $u^{(k)} \to u$ in $L^1(\Omega_T,\mathbb{R}^N)$ as $k\to\infty$. The lower semi-continuity of $\mathbf{F}$ with respect to weak convergence in $W^{1,p}(\Omega,\mathbb{R}^N)$ implies that

$$\mathbf{F}(u(t)) \le \liminf_{\mathcal{K}_1(t)\ni k\to\infty} \mathbf{F}(u_k(t)) = \liminf_{k\to\infty} \mathbf{F}(u_k(t)).$$

At this point, the claimed inequality follows with (3.1). □

Remark 3.2. The assumptions in the preceding lemma can be weakened in the sense that the strong L^1-convergence on Ω_T can be replaced by $u_k \to u$ strongly in $L^1(\Omega \times (\varepsilon,T],\mathbb{R}^N)$ for any $\varepsilon \in (0,T)$. □

We will frequently use a well known time regularization. For $v \in L^1(\Omega_T,\mathbb{R}^N)$, $v_o \in L^1(\Omega,\mathbb{R}^N)$ and $h>0$ we define

$$[v]_h(t) := \mathrm{e}^{-\frac{t}{h}} v_o + \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} v(s)\,\mathrm{d}s. \tag{3.2}$$

A straight-forward computation shows that

$$\partial_t [v]_h = -\tfrac{1}{h}\big([v]_h - v\big). \tag{3.3}$$

For more properties of the mollification we refer to [9, 27]. The next lemma ensures that $\mathbf{F}$ is continuous with respect to the time regularization $[v]_h$.

Lemma 3.3. *Let $p>1$ and $u_* \in L^2(\Omega,\mathbb{R}^N)\cap W^{1,p}(\Omega,\mathbb{R}^N)$, and assume that $\mathbf{F} : W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ is coercive in $u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$, sequentially lower semi-continuous with respect to the weak topology in $u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$, and convex on $L^2(\Omega,\mathbb{R}^N)\cap u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$. Consider*

$$v \in L^2(\Omega_T,\mathbb{R}^N)\cap L^p\big(0,T;u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)\big), \quad \text{with } \mathbf{F}(v)\in L^1(0,T)$$

and $v_o \in L^2(\Omega,\mathbb{R}^N)\cap u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$ with $\mathbf{F}(v_o)<\infty$. Then, for any $t\in[0,T]$ we have

$$\mathbf{F}\big([v]_h(t)\big) \le [\mathbf{F}(v)]_h(t),$$

where $[\mathbf{F}(v)]_h$ is defined according to definition (3.2) with v_o replaced by $\mathbf{F}(v_o)$. Moreover, $\mathbf{F}([v]_h) \in L^1(0,T)$ and

$$\lim_{h\downarrow 0}\int_0^T \mathbf{F}\big([v]_h\big)\,\mathrm{d}t = \int_0^T \mathbf{F}(v)\,\mathrm{d}t.$$

Proof. We first observe that

$$\frac{1}{h(1-e^{-\frac{t}{h}})}\int_0^t e^{\frac{s-t}{h}}\,\mathrm{d}s = 1.$$

This allows us to interpret the mollification $[v]_h$ – modulo a multiplicative factor – as a mean with respect to the measure $e^{\frac{s-t}{h}}\mathrm{d}s$. Accordingly to this interpretation we first rewrite $\mathbf{F}([v]_h)$ and afterwards use the convexity of $\mathbf{F}$ and Jensen's inequality. This procedure yields the following point wise bound:

$$\begin{aligned}\mathbf{F}\big([v]_h(t)\big) &= \mathbf{F}\bigg(e^{-\frac{t}{h}}v_o + \frac{1-e^{-\frac{t}{h}}}{h(1-e^{-\frac{t}{h}})}\int_0^t v(s)e^{\frac{s-t}{h}}\,\mathrm{d}s\bigg)\\ &\le e^{-\frac{t}{h}}\mathbf{F}(v_o) + \big(1-e^{-\frac{t}{h}}\big)\,\mathbf{F}\bigg(\frac{1}{h(1-e^{-\frac{t}{h}})}\int_0^t v(s)e^{\frac{s-t}{h}}\,\mathrm{d}s\bigg)\\ &\le e^{-\frac{t}{h}}\mathbf{F}(v_o) + \tfrac{1}{h}\int_0^t \mathbf{F}(v(s))e^{\frac{s-t}{h}}\,\mathrm{d}s\\ &= [\mathbf{F}(v)]_h(t).\end{aligned}$$

Since $\mathbf{F}(v) \in L^1(0,T)$ and $\mathbf{F}(v_o) < \infty$ by assumption, an elementary calculation yields the uniform bound

$$\int_0^T [\mathbf{F}(v)]_h\,\mathrm{d}t \le \int_0^T \mathbf{F}(v)\,\mathrm{d}t + h\,\mathbf{F}(v_o) < \infty,$$

which proves $\mathbf{F}([v]_h) \in L^1(0,T)$. Moreover, from well-known properties of the time mollification, we infer $[v]_h \to v$ in $L^1(\Omega_T,\mathbb{R}^N)$ as $h \downarrow 0$, so that Lemma 3.1 is applicable. Combining this lemma with the preceding inequalities, we find that

$$\int_0^T \mathbf{F}(v)\,\mathrm{d}t \le \liminf_{h\downarrow 0}\int_0^T \mathbf{F}\big([v]_h(t)\big)\,\mathrm{d}t \le \limsup_{h\downarrow 0}\int_0^T [\mathbf{F}(v)]_h\,\mathrm{d}t \le \int_0^T \mathbf{F}(v)\,\mathrm{d}t.$$

This proves the last claim of the lemma. □

The next statement ensures the existence of the time derivative in $L^2(\Omega_R,\mathbb{R}^N)$, provided the initial values coincide with the lateral boundary values.

Lemma 3.4. *Let $p > 1$ and assume that u_* satisfies (2.1) and $\mathbf{F} : W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ is coercive in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$, sequentially lower semi-continuous with respect to the weak topology in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$, and convex on $L^2(\Omega,\mathbb{R}^N) \cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$. Then, any variational solution u to the gradient flow associated to $\mathbf{F}$ in the sense of Definition 2.1 with initial values $u_o = u_*$ satisfies $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ and $Du \in L^\infty(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$.*

Proof. Using u_* as comparison map in the variational inequality (2.3), we observe that the assumption $\mathbf{F}(u_*) < \infty$ from (2.1) implies $\mathbf{F}(u) \in L^1(\Omega_T)$. Next, we choose the mollification in time $[u]_h$ defined in (3.2) with $v_o = u_*$ as comparison function in (2.3). Since $[u]_h(0) = u_* = u_o$, we obtain

$$\int_0^\tau \mathbf{F}(u)\,\mathrm{d}t \le \int_0^\tau \mathbf{F}\big([u]_h\big)\,\mathrm{d}t + \iint_{\Omega_\tau} \partial_t[u]_h\cdot\big([u]_h - u\big)\,\mathrm{d}x\mathrm{d}t,$$

for any $\tau \in (0,T]$. Due to (3.3) and Lemma 3.3 this implies

$$\begin{aligned}\iint_{\Omega_\tau} \big|\partial_t[u]_h\big|^2\mathrm{d}x\mathrm{d}t &= -\tfrac{1}{h}\iint_{\Omega_\tau} \partial_t[u]_h\cdot\big([u]_h - u\big)\mathrm{d}x\mathrm{d}t\\ &\le \tfrac{1}{h}\int_0^\tau \big[\mathbf{F}\big([u]_h\big) - \mathbf{F}(u)\big]\,\mathrm{d}t\\ &\le \tfrac{1}{h}\int_0^\tau \big[[\mathbf{F}(u)]_h - \mathbf{F}(u)\big]\,\mathrm{d}t\\ &= -\int_0^\tau \partial_t[\mathbf{F}(u)]_h\,\mathrm{d}t\\ &= \mathbf{F}(u_*) - [\mathbf{F}(u)]_h(\tau)\\ &\le \mathbf{F}(u_*) < \infty,\end{aligned}$$

where $[\mathbf{F}(u)]_h$ is defined according to (3.2) with v_o replaced by $\mathbf{F}(u_*)$. This ensures that the time derivative $\partial_t u$ exists with $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ together with the quantitative estimate

$$\iint_{\Omega_T} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \le \mathbf{F}(u_*).$$

Moreover, since $\mathbf{F}$ is nonnegative, we deduce from the preceding chain of inequalities

$$[\mathbf{F}(u)]_h(\tau) \le \mathbf{F}(u_*)$$

for any $\tau \in (0,T]$ and every $h \in (0,T]$. Letting $h \downarrow 0$ and using the coercivity of $\mathbf{F}$, we conclude the second claim $Du \in L^\infty(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$. This finishes the proof of the lemma. □

4. Convex integrals under the Legendre-Hadamard condition

In this section we consider some notions specific for systems of differential equations, as well as for the vector-valued calculus of variations, such as for instance *quasiconvexity*, *polyconvexity*, *null Lagrangians* (or *quasiaffinity*). We refer to the original work by Morrey [38], to Ball [4] and to the book of Dacorogna [14]. Specifically we are interested in conditions (either necessary or sufficient conditions) for the *convexity* of the integral

$$\mathbf{F}(u) = \int_\Omega f(x,u,Du)\,\mathrm{d}x\,, \quad \text{with } u \in u_* + W_0^{1,p}(\Omega,\mathbb{R}^N),$$

when u_* is a fixed function in $W^{1,p}(\Omega,\mathbb{R}^N)$ and therefore u has the same boundary values as u_*. In particular, we consider integrands whose second derivatives

$$\Big(\frac{\partial^2 f}{\partial \xi_i^\alpha \partial \xi_j^\beta}\Big)$$

satisfy the Legendre-Hadamard rank-one condition in the following sense.

Definition 4.1. We consider coefficient functions $a_{ij}^{\alpha\beta}\colon \Omega \to \mathbb{R}$ for $\alpha,\beta = 1,2,\dots,N$ and $i,j = 1,2,\dots,n$. We say that $(a_{ij}^{\alpha\beta}(x))$ satisfies the *Legendre-Hadamard condition* iff

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x)\theta^\alpha\theta^\beta\eta_i\eta_j \ge 0 \tag{4.1}$$

holds true for every $x \in \Omega$ and every $\theta \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$. We say that $(a_{ij}^{\alpha\beta}(x))$ satisfies the *strict Legendre-Hadamard condition* iff

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x)\theta^\alpha\theta^\beta\eta_i\eta_j \ge \nu|\theta|^2|\eta|^2 \tag{4.2}$$

holds true for a constant $\nu > 0$, every $x \in \Omega$ and every $\theta \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$.

Prior work to the present article can be found in [13, 15, 16], and [14, first edition]. They concern the scalar setting and mostly the case $N = n = 1$.

4.1. The case without lower order terms.

We first discuss the special case where $f \in C^2(\mathbb{R}^{N\times n})$, and

$$\mathbf{F}(u) = \int_\Omega f(Du(x))\mathrm{d}x, \quad \text{with } u \in u_* + W_0^{1,\infty}(\Omega,\mathbb{R}^N).$$

We have the following relationship between integral convexity and rank one convexity.

Theorem 4.2.

(i) *If $\mathbf{F}$ is convex on $u_* + W_0^{1,\infty}(\Omega,\mathbb{R}^N)$, then f is rank one convex.*

(ii) *If f is quadratic and rank one convex, then* $\mathbf{F}$ *is convex on* $u_* + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$.

Proof. The convexity of $\mathbf{F}$ is clearly equivalent to the convexity of

$$g(t) = \mathbf{F}(u + t\varphi) = \int_\Omega f\big(Du(x) + tD\varphi(x)\big)\mathrm{d}x,$$

where $u \in u_* + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ and $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$. Since our hypotheses imply $g \in C^2(\mathbb{R})$, the convexity of $\mathbf{F}$ is equivalent to

$$\begin{aligned} 0 \le g''(0) &= \int_\Omega \big\langle D^2 f(Du(x))D\varphi(x), D\varphi(x)\big\rangle \mathrm{d}x \\ &= \int_\Omega \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial^2 f}{\partial \xi_i^\alpha \partial \xi_j^\beta}(Du(x))\, \varphi_{x_i}^\alpha \varphi_{x_j}^\beta \,\mathrm{d}x. \end{aligned} \tag{4.3}$$

We first prove assertion (i). To this aim we choose $x_o \in \Omega$ and $\varrho > 0$ sufficiently small so that $B_\varrho(x_o) \Subset \Omega$ and then φ with $\operatorname{spt}\varphi \Subset B_\varrho(x_o)$. For $\xi \in \mathbb{R}^{N\times n}$ we choose any $u \in u_* + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ so that $Du = \xi$ in $B_\varrho(x_o)$. Since $\mathbf{F}$ is convex, we thus deduce that

$$\int_{B_\varrho(x_o)} \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial^2 f}{\partial \xi_i^\alpha \partial \xi_j^\beta}(\xi)\varphi_{x_i}^\alpha \varphi_{x_j}^\beta \,\mathrm{d}x \ge 0$$

for any $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ with $\operatorname{spt}\varphi \Subset B_\varrho(x_o)$. Using Fourier transform (cf. [14, Lemma 5.27]), we deduce the Legendre-Hadamard condition (4.1) for the second derivatives $\big(\frac{\partial^2 f}{\partial \xi_i^\alpha \partial \xi_j^\beta}(\xi)\big)$ in the sense of Definition 4.1, showing that indeed f is rank one convex.

For the proof of assertion (ii) we may assume that f is quadratic and rank one convex. In particular, $D^2 f$ is constant. Therefore, using Fourier transform, the computation in (4.3) implies the convexity of $\mathbf{F}$. □

Remark 4.3.

(i) In the scalar case $N = 1$, rank one convexity and convexity are equivalent and the theorem is then stronger, since the convexity of f is then equivalent to the convexity of $\mathbf{F}$.

(ii) Note that the convexity of $\mathbf{F}$ does not imply the polyconvexity of f. Indeed if $n, N \ge 3$ and f is quadratic, there are examples (see [14, Thm. 5.25]) of quadratic functions that are rank one convex but not polyconvex. This shows, in particular, that the convexity of $\mathbf{F}$ does not imply that f is a sum of a convex function and a quasiaffine function (also called null-Lagrangian). However when $n = 2$ or $N = 2$ and f is quadratic, the convexity of $\mathbf{F}$ implies that f is indeed a sum of a convex and a quasiaffine function, see [32, 41, 42].

(iii) We also give below a non-quadratic example (cf. Proposition 4.4) showing that, even when $N = n = 2$, polyconvexity and therefore rank one convexity does not imply, in general, that $\mathbf{F}$ is convex.

(iv) If $p \ge 2$ and
$$|D^2 f(\xi)| \le a + b|\xi|^{p-2}$$
we can replace the affine space $u_* + W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ by $u_* + W_0^{1,p}(\Omega, \mathbb{R}^N)$, as for example in the quadratic case where $p = 2$.

We now turn to a non-quadratic example.

Proposition 4.4. *Let $n = N = 2$ and $f(\xi) = (\det \xi)^2$. Then* $\mathbf{F}$ *is not convex over* $W_0^{1,\infty}(\Omega, \mathbb{R}^2)$.

Proof. Before starting with the proof, let us introduce the following notation. For

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

we let

$$\widetilde{\xi} = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix},$$

so that

$$\det(\xi + \eta) = \det \xi + \langle \widetilde{\xi}, \eta \rangle + \det \eta.$$

We then let $\Omega = (0, 2\pi) \times (0, 2\pi)$ and consider

$$g(t) = \mathbf{F}(u + t\varphi) = \int_\Omega \big(\det(Du + tD\varphi)\big)^2 \mathrm{d}x.$$

for $u, \varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$. As in the proof of Theorem 4.2, we compute

$$g''(0) = 2 \int_\Omega \Big[\big(\langle \widetilde{Du}, D\varphi \rangle\big)^2 + 2 \det Du \det D\varphi \Big] \mathrm{d}x.$$

Choosing

$$u(x) = (\sin x_1, \sin x_2) \quad \text{and} \quad \varphi(x) = (\sin x_2, \sin x_1),$$

we find that $g''(0) < 0$. This shows that $\mathbf{F}$ is not convex. □

4.2. **Quadratic forms and the Gårding inequality.** In this section we consider the quadratic form $\mathbf{Q}\colon u_* + W_0^{1,2}(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathbf{Q}(u) = \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x) u_{x_i}^\alpha u_{x_j}^\beta \,\mathrm{d}x\,. \tag{4.4}$$

The following lemma is inspired by Gårding's inequality, cf. [23, § 1.1] and [37, § 6.5].

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $a_{ij}^{\alpha\beta} \in C^0(\overline{\Omega})$ for $i, j = 1, 2, \ldots, n$ and $\alpha, \beta = 1, 2, \ldots, N$ satisfy the strict Legendre-Hadamard condition (4.2). Then there exists a constant $c \geq 0$ such that $\mathbf{Q}(u) + c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}$ is a convex functional on the affine space $u_* + W_0^{1,2}(\Omega, \mathbb{R}^N)$ and also coercive, in the sense that*

$$\mathbf{Q}(u) + c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)} \geq \tfrac{1}{2}\nu \|Du\|^2_{L^2(\Omega,\mathbb{R}^{N\times n})} \tag{4.5}$$

for every $u \in u_* + W_0^{1,2}(\Omega, \mathbb{R}^N)$. The constant c depends on $n, N, \nu, \Omega, \|a_{ij}^{\alpha\beta}\|_{L^\infty}$, and on the modulus of continuity of the coefficients $a_{ij}^{\alpha\beta}$.

Proof. The proof is divided into six steps.

Step 1: The convexity of the functional

$$\mathbf{G}(u) = \mathbf{Q}(u) + c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}$$

is reduced to the convexity of the real function $g\colon [0, 1] \to \mathbb{R}$ defined by

$$g(t) = \mathbf{G}\big(tu_1 + (1 - t)u_2\big) = \mathbf{G}\big(u_2 + t(u_1 - u_2)\big),$$

for every $u_1, u_2 \in u_* + W_0^{1,2}(\Omega, \mathbb{R}^N)$. We compute the second derivative g'' of g at $t = 0$, and check whether or not $g''(0) \geq 0$. With the abbreviation $v = u_1 - u_2 \in W_0^{1,2}(\Omega, \mathbb{R}^N)$, this condition can be rewritten as

$$g''(0) = 2 \int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x) v_{x_i}^\alpha v_{x_j}^\beta \,\mathrm{d}x + 2c \int_\Omega |v|^2 \,\mathrm{d}x \geq 0. \tag{4.6}$$

Note, that this property is equivalent to the convexity of $\mathbf{G}$. The condition should hold for every $v \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Step 2: To make the presentation clearer, we shall use the more compact abbreviation

$$A(x) = \big(a_{ij}^{\alpha\beta}(x)\big).$$

Note that $A \in C^0\big(\overline{\Omega}, \mathbb{R}^{N\times n}\big)$. Moreover, for $\xi \in \mathbb{R}^{N\times n}$ we write

$$\big\langle A(x)\xi, \xi\big\rangle := \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta .$$

Now, since $\overline{\Omega}$ is a compact set in $\mathbb{R}^n$ and A continuous on $\overline{\Omega}$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|A(x) - A(x')| < \varepsilon, \text{ whenever } |x - x'| < \delta. \tag{4.7}$$

Moreover, there exists a finite set of points x_s in Ω, $s = 1, 2, \ldots, S$, and open balls $B_\delta(x_s)$ centered in x_s with radius δ, such that

$$\Omega \subset \bigcup_{s=1}^{S} B_\delta(x_s).$$

For every index $s = 1, 2, \ldots, S$ we find test functions $\eta_s \in C_0^\infty(B_\delta(x_s))$ with $0 \le \eta_s \le 1$ and $|D\eta_s| \le M_\eta$ for a constant M_η independent of s, such that

$$\sum_{s=1}^{S} \big(\eta_s(x)\big)^2 = 1 \qquad \forall\, x \in \Omega,$$

i.e. the family $(\eta_s)^2$ forms a partition of unity subordinate to the covering $(B_\delta(x_s))$. We extend $v \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ to be equal to zero outside of Ω. We then have

$$\int_\Omega \langle ADv, Dv\rangle \,\mathrm{d}x = \sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle A\eta_s Dv, \eta_s Dv\big\rangle \,\mathrm{d}x. \tag{4.8}$$

Step 3: We now compute the gradient

$$D(\eta_s v) = \eta_s Dv + v \otimes D\eta_s \tag{4.9}$$

and thus from (4.8) we obtain

$$\begin{aligned}\int_\Omega \big\langle ADv, Dv\big\rangle \,\mathrm{d}x &= \sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle A\big(D(\eta_s v) - v \otimes D\eta_s\big), D(\eta_s v) - v \otimes D\eta_s\big\rangle \,\mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},\end{aligned} \tag{4.10}$$

with the obvious abbreviations

$$\begin{aligned}\mathrm{I} &:= \sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle AD(\eta_s v), D(\eta_s v)\big\rangle \,\mathrm{d}x,\\ \mathrm{II} &:= -\sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle Av \otimes D\eta_s, D(\eta_s v)\big\rangle \,\mathrm{d}x,\\ \mathrm{III} &:= -\sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle AD(\eta_s v), v \otimes D\eta_s\big\rangle \,\mathrm{d}x,\\ \mathrm{IV} &:= \sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle Av \otimes D\eta_s, v \otimes D\eta_s\big\rangle \,\mathrm{d}x.\end{aligned}$$

In the next steps we estimate separately the four addenda.

Step 4: We let $M_A := \|A\|_{L^\infty(\Omega,\mathbb{R}^{N\times n})}$, so that $|A(x)| \le M_A$ for all $x \in \overline{\Omega}$. For the last addendum in (4.10) we have

$$|\mathrm{IV}| \le M_A \|v\|_{L^2(\Omega,\mathbb{R}^N)}^2 \sum_{s=1}^{S} \|D\eta_s\|_{L^\infty(\Omega,\mathbb{R}^n)}^2 \le S M_A M_\eta^2 \|v\|_{L^2(\Omega,\mathbb{R}^N)}^2. \tag{4.11}$$

The second and the third addenda in (4.10) can be estimated similarly. Indeed, we have

$$|\mathrm{II}| + |\mathrm{III}| \le 2\sqrt{S} M_A M_\eta \|v\|_{L^2(\Omega,\mathbb{R}^N)} \Bigg[\sum_{s=1}^{S} \|D(\eta_s v)\|^2_{L^2(B_\delta(x_s),\mathbb{R}^N)} \Bigg]^{\frac12}.$$

With (4.9), we obtain

$$\|D(\eta_s v)\|^2_{L^2(B_\delta(x_s),\mathbb{R}^N)} \le 2 \int_{B_\delta(x_s)} \eta_s^2 |Dv|^2 \,\mathrm{d}x + 2M_\eta^2 \int_{B_\delta(x_s)} |v|^2 \,\mathrm{d}x.$$

Summing over $s = 1, \dots, S$ and taking into account that $(\eta_s^2)_{s=1,\dots,S}$ forms a partition of unity we obtain that

$$\sum_{s=1}^{S} \|D(\eta_s v)\|^2_{L^2(B_\delta(x_s),\mathbb{R}^N)} \le 2\|Dv\|^2_{L^2(\Omega,\mathbb{R}^{N\times n})} + 2SM_\eta^2 \|v\|^2_{L^2(\Omega,\mathbb{R}^N)}.$$

Inserting this above, we get for $\varepsilon_1 > 0$ that

$$\begin{aligned} |\mathrm{II}| + |\mathrm{III}| &\le \sqrt{8S} M_A M_\eta \|v\|_{L^2(\Omega,\mathbb{R}^N)} \Big[\|Dv\|^2_{L^2(\Omega,\mathbb{R}^{N\times n})} + SM_\eta^2 \|v\|^2_{L^2(\Omega,\mathbb{R}^N)} \Big]^{\frac12} \\ &\le \varepsilon_1 \|Dv\|^2_{L^2(\Omega,\mathbb{R}^{N\times n})} + \varepsilon_1 SM_\eta^2 \|v\|^2_{L^2(\Omega,\mathbb{R}^N)} + \frac{2SM_A^2 M_\eta^2}{\varepsilon_1} \|v\|^2_{L^2(\Omega,\mathbb{R}^N)}. \end{aligned} \tag{4.12}$$

For the first addendum in (4.10) we use the uniform continuity in (4.7). With the abbreviation

$$\mathrm{I}_s := \int_{B_\delta(x_s)} \big\langle A(x_s) D(\eta_s v), D(\eta_s v) \big\rangle \,\mathrm{d}x$$

we have

$$\begin{aligned} \mathrm{I} &= \sum_{s=1}^{S} \mathrm{I}_s + \sum_{s=1}^{S} \int_{B_\delta(x_s)} \big\langle (A(x) - A(x_s)) D(\eta_s v), D(\eta_s v) \big\rangle \,\mathrm{d}x \\ &\ge \sum_{s=1}^{S} \mathrm{I}_s - \varepsilon \sum_{s=1}^{S} \int_{B_\delta(x_s)} |D(\eta_s v)|^2 \,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Therefore it remains to estimate I_s from below. This we shall do in the next step.

Step 5: Here we make use of the Fourier transform, a tool already shown to be useful in the vector valued context of quadratic forms satisfying the Legendre-Hadamard condition (see [38], [14]). We start from the expression of the quadratic form with frozen coefficients for fixed $s \in \{1, 2, \dots, S\}$ and with $\eta_s v$ extended to be equal to zero out of the ball $B_\delta(x_s)$. Using Plancherel's formula twice and the strict Legendre-Hadamard condition (4.2), we obtain

$$\begin{aligned} \mathrm{I}_s &= \int_{\mathbb{R}^n} \big\langle A(x_s) \widehat{D(\eta_s v)}, \widehat{D(\eta_s v)} \big\rangle \,\mathrm{d}\xi = \int_{\mathbb{R}^n} \big\langle A(x_s) \widehat{\eta_s v} \otimes \xi, \widehat{\eta_s v} \otimes \xi \big\rangle \,\mathrm{d}\xi \\ &\ge \nu \int_{\mathbb{R}^n} \big| \widehat{\eta_s v} \otimes \xi \big|^2 \mathrm{d}\xi = \nu \int_{\mathbb{R}^n} \big| \widehat{D(\eta_s v)} \big|^2 \mathrm{d}\xi = \nu \int_{B_\delta(x_s)} |D(\eta_s v)|^2 \mathrm{d}x \end{aligned}$$

for every $s \in \{1, 2, \dots, S\}$. Joining this inequality with (4.13) and recalling that (η_s^2) is a partition of unity of Ω subordinate to the covering $(B_\delta(x_s))$, we obtain

$$\begin{aligned} \mathrm{I} &\ge (\nu - \varepsilon) \sum_{s=1}^{S} \int_{B_\delta(x_s)} |D(\eta_s v)|^2 \,\mathrm{d}x \\ &\ge \tfrac12 (\nu - \varepsilon) \sum_{s=1}^{S} \int_{B_\delta(x_s)} \eta_s^2 |Dv|^2 \mathrm{d}x - (\nu - \varepsilon) \sum_{s=1}^{S} \int_{B_\delta(x_s)} |v \otimes D\eta_s|^2 \mathrm{d}x \\ &\ge \tfrac12 (\nu - \varepsilon) \int_\Omega |Dv|^2 \mathrm{d}x - (\nu - \varepsilon) SM_\eta^2 \int_\Omega |v|^2 \mathrm{d}x. \end{aligned} \tag{4.14}$$

Step 6: We can conclude the proof of Lemma 4.5. Joining (4.10) with the estimates of the four addenda (4.11), (4.12), and (4.14), we obtain

$$\int_\Omega \langle ADv, Dv\rangle \,\mathrm{d}x \geq \left[\tfrac{1}{2}(\nu-\varepsilon)-\varepsilon_1\right]\int_\Omega |Dv|^2\mathrm{d}x - SM_\eta^2\left[M_A+\varepsilon_1+\frac{2M_A^2}{\varepsilon_1}+(\nu-\varepsilon)\right]\int_\Omega |v|^2\mathrm{d}x.$$

At this stage we choose ε and ε_1 small enough such that $\frac{1}{2}(\nu-\varepsilon)-\varepsilon_1 \geq \frac{1}{2}\nu$. These particular choices specify the constant in front of the second integral on the right-hand side. Overall, the constant depends on $n, N, \nu, \Omega, \|a_{ij}^{\alpha\beta}\|_{L^\infty}$, and the modulus of continuity of the coefficients $a_{ij}^{\alpha\beta}$, but it is independent of u. Altogether we have shown that

$$\int_\Omega \langle ADv, Dv\rangle \,\mathrm{d}x \geq \tfrac{1}{2}\nu\int_\Omega |Dv|^2\mathrm{d}x - c\int_\Omega |v|^2\mathrm{d}x,$$

so that the second derivative g'' in (4.6) is positive and the integral functional $\mathbf{Q}(u)+c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}$ is coercive as stated in (4.5). This concludes the proof of the Lemma. □

Now, Lemma 4.5 leads to the main theorem of this section.

Theorem 4.6. *Let Ω, $a_{ij}^{\alpha\beta}$ be as in Lemma 4.5. In particular we assume that the strict Legendre-Hadamard condition (4.2) is satisfied. Let $c^{\alpha\beta}\in L^\infty(\Omega)$ for every $\alpha,\beta=1,2,\ldots,N$. Furthermore, assume that*

$$\sum_{\alpha,\beta=1}^N c^{\alpha\beta}(x)u^\alpha u^\beta \geq c_o|u|^2\,, \quad \text{for any } u\in\mathbb{R}^N,$$

for a certain constant $c_o\geq 0$. Let

$$\mathbf{G}(u)=\int_\Omega\left[\sum_{i,j=1}^n\sum_{\alpha,\beta=1}^N a_{ij}^{\alpha\beta}(x)u^\alpha_{x_i}u^\beta_{x_j}+\sum_{\alpha,\beta=1}^N c^{\alpha\beta}(x)u^\alpha u^\beta\right]\mathrm{d}x\,.$$

Then for c_o large enough the functional $\mathbf{G}$ is convex and coercive on $u_*+W_0^{1,2}(\Omega,\mathbb{R}^N)$.

Proof. If $c_o\geq c$, where c is the constant from Lemma 4.5 the energy integral $\mathbf{G}$ is convex and coercive on $u_*+W_0^{1,2}(\Omega,\mathbb{R}^N)$. In fact, if $\mathbf{Q}$ is the quadratic form as in (4.4) and if we denote by I the identity $N\times N$ matrix, we have

$$\begin{aligned}\mathbf{G}(u)&=\mathbf{Q}(u)+\int_\Omega\langle C(x)u,u\rangle\,\mathrm{d}x\\&=\mathbf{Q}(u)+c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}+\int_\Omega\big\langle (C(x)-c\,I)u,u\big\rangle\,\mathrm{d}x,\end{aligned}$$

i.e. $\mathbf{G}(u)$ is the sum of the convex and coercive functional $\mathbf{Q}(u)+c\|u\|^2_{L^2(\Omega,\mathbb{R}^N)}$ and the integral $\int_\Omega\langle (C(x)-c\,I)u,u\rangle\,\mathrm{d}x$, which is a convex and nonnegative functional, since the matrix $(C(x)-cI)$ is positive semidefinite. □

4.3. **Linear parabolic systems under the Legendre-Hadamard condition.** Let $n,N\in\mathbb{N}$ with $n,N\geq 2$. We are interested in the existence of weak solutions to the Cauchy-Dirichlet problem associated to the *linear parabolic system of N partial differential equations*

$$\frac{\partial u^\alpha}{\partial t}-\sum_{i,j=1}^n\sum_{\beta=1}^N\frac{\partial}{\partial x_i}\Big(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\Big)-\sum_{\beta=1}^N c^{\alpha\beta}u^\beta=h^\alpha \qquad \text{in } \Omega_T \tag{4.15}$$

for $\alpha=1,2,\ldots,N$ with continuous coefficients $a_{ij}^{\alpha\beta}=a_{ij}^{\alpha\beta}(x)$, $c^{\alpha\beta}=c^{\alpha\beta}(x)$, for $\alpha,\beta=1,2,\ldots,N$ and $i,j=1,2,\ldots,n$. As before, $u=u(x,t)$ is a map defined on the parabolic cylinder Ω_T with values in $\mathbb{R}^N$. Finally, $h\in L^2(\Omega,\mathbb{R}^N)$ is a given right-hand side. We note that the system has a *variational structure*; i.e. in the stationary case when $u(x,t)$ is independent of $t>0$, the differential system (4.15) is the Euler-Lagrange system associated to the first variation of an energy functional as in (1.1). In the general nonlinear case the first variation takes the form

$$\operatorname{div}\big(D_\xi f(x,u,Du)\big) = D_u f(x,u,Du).$$

For the specific quadratic case considered in (4.15), the integrand f is given by

$$f(x,u,\xi) = \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x)\xi_i^\alpha\xi_j^\beta + \sum_{\alpha,\beta=1}^{N} c^{\alpha\beta}(x)u^\alpha u^\beta - \sum_{\beta=1}^{N} h^\beta(x)u^\beta.$$

At this stage, we need to explain in which sense the considered linear operator is *coercive*: here we consider the strict Legendre-Hadamard rank-one condition (4.2) as stated in Definition 4.1. We emphasize that (4.15) is a *linear parabolic system satisfying the strict Legendre-Hadamard rank-one condition*, since we do not assume the simpler and less general ellipticity condition

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x)\xi_i^\alpha\xi_j^\beta \ge \nu|\xi|^2, \quad \text{for every } \xi\in\mathbb{R}^{N\times n},$$

which guarantees existence of the solution to the Cauchy-Dirichlet problem associated to the parabolic system (4.15), cf. [21]. Both conditions are equivalent only the one-dimensional case $n=1$ and in the scalar case $N=1$.

The existence of stationary weak solutions to the Dirichlet problem associated to (4.15) under the strict Legendre-Hadamard condition (4.2) can be retrieved for example from [23]. A similar result for the Cauchy-Dirichlet problem associated to (4.15) can be obtained by applying Theorem 7.5 below. We only need to assume the strict Legendre-Hadamard condition (4.2) and the continuity $a_{ij}^{\alpha\beta}, c^{\alpha\beta}\in C^0(\overline{\Omega})$. Under these hypotheses the Cauchy-Dirichlet problem associated to (4.15) has a unique weak solution provided the zero-order terms $c^{\alpha\beta}(x)$ are sufficiently large in the sense that

$$\sum_{\alpha,\beta=1}^{N} c^{\alpha\beta}(x)u^\alpha u^\beta \ge c_o|u|^2\,,$$

holds true for every $x\in\Omega$ and for some sufficiently large constant $c_o\in\mathbb{R}$.

5. Integral convexity with non-convex integrands

In this section we consider a Carathéodory integrand $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfying for every $(x,u,\xi)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ the growth condition

$$|f(x,u,\xi)| \le m_0\big(1+|\xi|^2+|u|^q\big) \tag{5.1}$$

for some constant $m_0>0$ and some exponent $q\ge 2$. We note that we do not require that q remains below the Sobolev exponent 2^*. Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^n$. Under the above growth condition, the associated variational integral $\mathbf{F}$ from (1.1) is well defined for any $u\in W^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$. The main purpose of this section is to find suitable conditions on the integrand f under which the integral functional $\mathbf{F}\colon u_o+W_0^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)\to\mathbb{R}$ is convex. Here, $u_o\in W^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$ denotes some fixed Dirichlet boundary value. Of course, this question can be reduced to the study of the real-valued function $g\colon[0,1]\to\mathbb{R}$ defined by

$$g(t) = \mathbf{F}\big(tu_1+(1-t)u_2\big),$$

for every $u_1,u_2\in u_o+W_0^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$. In addition to the growth condition (5.1) we assume that the partial maps $\mathbb{R}^N\times\mathbb{R}^{N\times n}\ni(u,\xi)\mapsto f(x,u,\xi)$ are twice continuously differentiable for a.e. $x\in\Omega$, and that the second derivatives $D_u^2 f(x,u,\xi)$,

$D_uD_\xi f(x,u,\xi)$, $D^2_\xi f(x,u,\xi)$ remain bounded as long as (x,u,ξ) stay in bounded subsets of $\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$. Note that the second order derivatives exist for a.e. $x\in\Omega$. As mappings defined on $\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ they are also Carathéodory functions.

In order to establish the convexity of the functional $\mathbf{F}$ on the affine space $u_o + W^{1,2}_0(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$ we first consider the special case when $u_o\in W^{1,\infty}(\Omega,\mathbb{R}^N)$ and $u_1,u_2\in u_o+W^{1,\infty}_0(\Omega,\mathbb{R}^N)$. To simplify the notation we write u instead of u_1 and v instead of u_2-u_1. Note that $v\in W^{1,\infty}_0(\Omega,\mathbb{R}^N)$. Under these conditions, the real-valued function

$$g(t)=\int_\Omega f\big(x,u+tv,Du+tDv\big)\,\mathrm{d}x$$

is twice differentiable. This can be justified by considering difference quotients. The boundedness of the second derivatives of f on bounded subsets allows us to pass to the limit inside the integral by using Lebesgue's dominated convergence theorem. Therefore, differentiation under the integral is allowed. We omit the details and give only the outcome of this procedure. We compute

$$g'(t)=\int_\Omega\Big[D_uf\big(x,u+tv,Du+tDv\big)\cdot v+D_\xi f\big(x,u+tv,Du+tDv\big)Dv\Big]\,\mathrm{d}x,$$

and

$$\begin{aligned}g''(0)=\int_\Omega\Big[&\big\langle D^2_uf\big(x,u,Du\big)v,v\big\rangle+\big\langle D^2_\xi f\big(x,u,Du\big)Dv,Dv\big\rangle\\&+2\big\langle D_\xi D_uf\big(x,u,Du\big)Dv,v\big\rangle\Big]\,\mathrm{d}x.\end{aligned}$$

We now impose the following assumptions on the second derivatives of f:

$$\left\{\begin{aligned}\big\langle D^2_\xi f(x,u,\xi)\lambda,\lambda\big\rangle&\ge m_1|\lambda|^2\\ \big\langle D^2_uf(x,u,\xi)\theta,\theta\big\rangle&\ge -m_2|\theta|^2\\ \big|D_\xi D_uf(x,u,\xi)\big|&\le m_3\end{aligned}\right.\tag{5.2}$$

for every $x\in\Omega$, $u,\theta\in\mathbb{R}^N$ and $\xi,\lambda\in\mathbb{R}^{N\times n}$. For the constants appearing on the right-hand side we require that $m_1>0$, $m_2\in\mathbb{R}$ and $m_3\ge 0$. This gives

$$\begin{aligned}g''(0)&\ge\int_\Omega\Big[-m_2|v|^2-2m_3|Dv||v|+m_1|Dv|^2\Big]\mathrm{d}x\\&\ge\int_\Omega\Big[-(m_2+m_3)|v|^2+(m_1-m_3)|Dv|^2\Big]\mathrm{d}x\,.\end{aligned}$$

At this stage we recall the Poincaré inequality

$$\int_\Omega|v|^2\,\mathrm{d}x\le P_o\int_\Omega|Dv|^2\,\mathrm{d}x$$

which holds, because $v=u_1-u_2\in W^{1,\infty}_0(\Omega,\mathbb{R}^N)$. Here $P_o=P_o(n)$ denotes the *Poincaré constant*. For instance, in the two dimensional case $n=2$ it is known that $P_o\le\big(\frac{d}{\pi}\big)^2$, where d is the diameter of Ω. We therefore get

$$g''(0)\ge\Big[(m_1-m_3)-P_o(m_2+m_3)\Big]\int_\Omega|Dv|^2\,\mathrm{d}x\,.$$

Therefore, if we require

$$m_1\ge P_om_2+(1+P_o)m_3\,,\tag{5.3}$$

we have $g''(0)\ge 0$ and the functional $\mathbf{F}$ from (1.1) is convex on $u_o+W^{1,\infty}_0(\Omega,\mathbb{R}^N)$.

In the general case of mappings $u_o\in W^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$ and $u_1,u_2\in u_o+W^{1,2}_0(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$ we use a standard regularization procedure. In the strong topology of $W^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$, we approximate u_o by a sequence of maps $u^{(\varepsilon)}_o\in$

$W^{1,\infty}(\Omega,\mathbb{R}^N)$, and $u_i - u_o$, $i = 1,2$ by sequences $v_i^{(\varepsilon)} \in W_0^{1,\infty}(\Omega,\mathbb{R}^N)$. Then, we let $u_i^{(\varepsilon)} := u_o^{(\varepsilon)} + v_i^{(\varepsilon)}$ for $i = 1,2$. The convexity of $\mathbf{F}$ on the affine subspace $u_o^{(\varepsilon)} + W_0^{1,\infty}(\Omega,\mathbb{R}^N)$ implies

$$\mathbf{F}\big(tu_1^{(\varepsilon)} + (1-t)u_2^{(\varepsilon)}\big) \le t\,\mathbf{F}\big(u_1^{(\varepsilon)}\big) + (1-t)\mathbf{F}\big(u_2^{(\varepsilon)}\big). \tag{5.4}$$

Now, the growth condition (5.1) and the dominated convergence theorem together ensure the continuity of $\mathbf{F}$ in the strong topology of $W^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$, that is

$$\lim_{\varepsilon\downarrow 0}\mathbf{F}\big(u_i^{(\varepsilon)}\big) = \mathbf{F}(u_i) \quad \text{for } i = 1,2,$$

and also

$$\lim_{\varepsilon\downarrow 0}\mathbf{F}\big(tu_1^{(\varepsilon)} + (1-t)u_2^{(\varepsilon)}\big) = \mathbf{F}(tu_1 + (1-t)u_2).$$

Therefore, we can pass to the limit $\varepsilon \downarrow 0$ on both sides of (5.4) and obtain the validity of the convexity condition on $u_o + W_0^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)$. Altogether, we have proved the following result.

Theorem 5.1. *Let Ω be bounded domain in $\mathbb{R}^n$ and $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ be a Carathéodory integrand. Assume further, that the partial map $(u,\xi)\mapsto f(x,u,\xi)$ is twice continuously differentiable for a.e. $x\in\Omega$ and that the derivatives $D_\xi^2 f(x,u,\xi)$, $D_uD_\xi f(x,u,\xi)$, $D_u^2 f(x,u,\xi)$ are bounded on bounded subset of $\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$. Finally, we require that the growth and coercivity conditions* (5.1), (5.2) *and the smallness condition* (5.3) *are in force. Then the energy integral $\mathbf{F}\colon u_o + W_0^{1,2}(\Omega,\mathbb{R}^N)\cap L^q(\Omega,\mathbb{R}^N)\to\mathbb{R}$ in* (1.1) *is convex.*

At this point, one should emphasize that condition (5.3) holds, for instance, if m_2 is positive and sufficiently small and m_3 is equal to zero. The one-dimensional case with $m_3 = 0$ was already considered in [14, § 3.2.2 and Prop. 2.5]. An other example can be found in [40, § 8.1]. An energy integral (1.1) whose integrand $f = f(x,u,\xi)$ is not necessarily convex with respect to (u,ξ) and satisfies (5.2) is

$$\mathbf{F}(u) = \int_\Omega \Big[f(x,Du) + g(x,u) + h(x,u,Du)\Big]\,\mathrm{d}x\,,$$

where $\big\langle D_\xi^2 f(x,\xi)\lambda,\lambda\big\rangle \ge m_1|\lambda|^2$ and $\big\langle D_u^2 g(x,u)\theta,\theta\big\rangle \ge -m_2|\theta|^2$, while the integrand h is of the form

$$h(x,u,\xi) := \sum_{i=1}^n\sum_{\alpha=1}^N c_i(x)u^\alpha\xi_i^\alpha \equiv \langle u\otimes c(x),\xi\rangle$$

with $c\in L^\infty(\Omega,\mathbb{R}^n)$. Then, $|D_uD_\xi h(x,u,\xi)| \le \|c\|_{L^\infty(\Omega,\mathbb{R}^n)} =: m_3$. Assuming that $m_1 > 0$, $m_2\in\mathbb{R}$ and $m_3 \ge 0$ satisfy (5.3), then the hypotheses of Theorem 5.1 are satisfied. In particular, for the integrand

$$g(x,u) := b(x)\big(|u|^2-1\big)^2,$$

with a nonnegative function $b\in L^\infty(\Omega)$ we have

$$\big\langle D_u^2 g(x,u)\theta,\theta\big\rangle = 4\,b(x)\Big[2(u\cdot\theta)^2 + \big(|u|^2-1\big)|\theta|^2\Big] \ge -4\|b\|_{L^\infty(\Omega)}|\theta|^2$$

so that $(5.2)_2$ is satisfied with $m_2 = 4\|b\|_{L^\infty(\Omega)}$. Therefore, the integral

$$\mathbf{F}(u) = \int_\Omega \Big[f(x,Du) + b(x)\big(|u|^2-1\big)^2 + u\cdot(Du\,c)\Big]\,\mathrm{d}x \tag{5.5}$$

is convex on the affine subspace $u\in u_o + W_0^{1,2}(\Omega,\mathbb{R}^N)\cap L^4(\Omega,\mathbb{R}^N)$ provided we assume that $b\ge 0$, that $\big\langle D_\xi^2 f(x,\xi)\lambda,\lambda\big\rangle \ge m_1|\lambda|^2$, and that

$$m_1 \ge 4P_o\|b\|_{L^\infty(\Omega)} + (1+P_o)\|c\|_{L^\infty(\Omega,\mathbb{R}^n)}\,.$$

6. Integral convexity as necessary condition for existence

Here we prove that the convexity of $\mathbf{F}$ is a necessary condition for the existence of variational solutions. In particular, the convexity of the integral functional can not be weakened to quasiconvexity of the integrand. This seems to be a fundamental difference between elliptic and evolutionary variational problems. For integrands with quadratic growth this fact has already been observed by Wieser [43, Thm. 5.1]. For a more recent result we also refer to [17]. Here we give Wieser's result in a more general setting with a completely different proof. With our proof we are able to treat the case $p \neq 2$ and general functionals without growth conditions from above. We point out that the following result is slightly stronger than the necessity part of Theorem 2.3 in the sense that we only need to assume existence for initial values $u_o \in L^2(\Omega,\mathbb{R}^N)\cap W^{1,p}(\Omega,\mathbb{R}^N)$ and not for any $u_o \in L^2(\Omega,\mathbb{R}^N)$. Moreover, it suffices to assume that $\mathbf{F}$ is coercive in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$ instead of the stronger assumption $(2.2)_1$.

Theorem 6.1. *Let $p > 1$, $u_* \in W^{1,p}(\Omega,\mathbb{R}^N)$ and $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ be a variational functional which is coercive and sequentially weakly lower semi-continuous on $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$. Suppose that for every initial datum $u_o \in L^2(\Omega,\mathbb{R}^N) \cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$ there exists a variational solution $u \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;u_* + W^{1,p}_0(\Omega,\mathbb{R}^N))$ to the gradient flow for $\mathbf{F}$ in the sense of Definition 2.1. Then $\mathbf{F}$ is convex on $L^2(\Omega,\mathbb{R}^N) \cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$.*

Proof. We fix $w_0, w_1 \in L^2(\Omega,\mathbb{R}^N)\cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$ with $\mathbf{F}(w_i) < \infty$ for $i \in \{1,2\}$. For $\lambda \in [0,1]$, we consider the convex combination $w_\lambda := (1-\lambda)w_0 + \lambda w_1 \in L^2(\Omega,\mathbb{R}^N)\cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$ as initial values. According to the above assumptions, there exists a variational solution $u \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;u_* + W^{1,p}_0(\Omega,\mathbb{R}^N))$ to the gradient flow for $\mathbf{F}$ with the initial datum w_λ. We exploit the variational inequality (2.3) for this solution u once with the comparison map $v(x,t) = w_0(x)$ and once with $v(x,t) = w_1(x)$, which yields the inequalities

$$
\begin{aligned}
\int_0^\tau \mathbf{F}(u)\,\mathrm{d}t &\le \tau\,\mathbf{F}(w_0) - \tfrac12\|w_0 - u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12\|w_0 - w_\lambda\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&= \tau\,\mathbf{F}(w_0) - \langle w_0, w_\lambda - u(\tau)\rangle_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad + \tfrac12\|w_\lambda\|^2_{L^2(\Omega,\mathbb{R}^N)} - \tfrac12\|u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\tau \mathbf{F}(u)\,\mathrm{d}t &\le \tau\,\mathbf{F}(w_1) - \tfrac12\|w_1 - u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12\|w_1 - w_\lambda\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&= \tau\,\mathbf{F}(w_1) - \langle w_1, w_\lambda - u(\tau)\rangle_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad + \tfrac12\|w_\lambda\|^2_{L^2(\Omega,\mathbb{R}^N)} - \tfrac12\|u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)},
\end{aligned}
$$

both for a.e. $\tau \in [0,T]$. We multiply the first inequality with $(1-\lambda)/\tau$, the second one with λ/τ, and sum up the results. This leads us to

$$
\begin{aligned}
⨍_0^\tau \mathbf{F}(u)\,\mathrm{d}t &\le (1-\lambda)\mathbf{F}(w_0) + \lambda\mathbf{F}(w_1) \\
&\qquad - \tfrac1\tau\Big[\langle w_\lambda, w_\lambda - u(\tau)\rangle_{L^2(\Omega,\mathbb{R}^N)} - \tfrac12\|w_\lambda\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12\|u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)}\Big] \\
&= (1-\lambda)\mathbf{F}(w_0) + \lambda\mathbf{F}(w_1) - \tfrac{1}{2\tau}\|w_\lambda - u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\le (1-\lambda)\mathbf{F}(w_0) + \lambda\mathbf{F}(w_1)
\end{aligned}
\tag{6.1}
$$

for every $\tau \in A_1$, for a subset $A_1 \subset [0,T]$ with $|A_1| = T$. From Remark 2.2 we know that $u(t) \to u(0) = w_\lambda$ strongly in $L^2(\Omega,\mathbb{R}^N)$ as $t \downarrow 0$. Next, we consider the set $A_2 \subset [0,T]$

of times $t \in [0,T]$ with $u(t) \in W^{1,p}(\Omega,\mathbb{R}^N)$, which satisfies $|A_2| = T$, and claim that

$$\mathbf{F}(w_\lambda) \le \liminf_{A_2 \ni t \downarrow 0} \mathbf{F}(u(t)). \tag{6.2}$$

If the right-hand side is infinite, there is nothing to prove. In the other case, we choose a sequence $A_2 \ni t_i \downarrow 0$ for which the limes inferior is attained, i.e.

$$\lim_{i\to\infty} \mathbf{F}(u(t_i)) = \liminf_{A_2 \ni t \downarrow 0} \mathbf{F}(u(t)) < \infty.$$

Since the functional $\mathbf{F}$ is coercive in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$, we deduce that the sequence $u(t_i)$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^N)$. By passing to a not relabelled subsequence, we can therefore assume that $u(t_i) \rightharpoondown v$ weakly in $W^{1,p}(\Omega,\mathbb{R}^N)$ in the limit $i \to \infty$, for some $v \in u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$. Since we know already that $u(t_i) \to w_\lambda$ in $L^2(\Omega,\mathbb{R}^N)$, we conclude $v = w_\lambda$. Now, the claim (6.2) is a consequence of the lower semi-continuity of $\mathbf{F}$ with respect to weak convergence in $u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$. From (6.2) we deduce furthermore that

$$\mathbf{F}(w_\lambda) \le \liminf_{A_2 \ni t \downarrow 0} \mathbf{F}(u(t)) \le \liminf_{\tau\downarrow 0} \fint_0^\tau \mathbf{F}(u(t))\,\mathrm{d}t \le \liminf_{A_1 \ni \tau \downarrow 0} \fint_0^\tau \mathbf{F}(u(t))\,\mathrm{d}t \tag{6.3}$$

holds true. Combining the estimates (6.1) and (6.3), we deduce

$$\mathbf{F}\big((1-\lambda)w_0 + \lambda w_1\big) = \mathbf{F}(w_\lambda) \le (1-\lambda)\mathbf{F}(w_0) + \lambda \mathbf{F}(w_1),$$

which yields the claimed convexity of $\mathbf{F}$ in the case $\mathbf{F}(w_i) < \infty$ for $i \in \{1,2\}$. If either $\mathbf{F}(w_0)$ or $\mathbf{F}(w_1)$ is infinite, the inequality holds trivially. This concludes the proof of the theorem. □

7. Integral convexity as sufficient condition for existence

7.1. **Proof via Elliptic Regularization.** In this subsection we prove the sufficiency part of Theorem 2.3 in the special case $u_o = u_*$.

Theorem 7.1. *Assume that $p > 1$ and that u_* and $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ satisfy (2.1) and (2.2)$_{1,2}$, and that $\mathbf{F}$ is convex on $L^2(\Omega,\mathbb{R}^N) \cap u_* + W^{1,p}_0(\Omega,\mathbb{R}^N)$. Then, there exists a variational solution $u \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^\infty(0,T;u_* + W^{1,p}_0(\Omega,\mathbb{R}^N))$ to the gradient flow for $\mathbf{F}$ in the sense of Definition 2.1 with initial datum $u_o = u_*$.*

7.1.1. *Elliptic Regularization.* We now consider for $\varepsilon \in (0,1]$ variational integrals on the space-time cylinder $\Omega_T := \Omega \times (0,T)$ defined by

$$\boldsymbol{\mathcal{F}}_\varepsilon(v) := \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \bigg[\int_\Omega \tfrac12 |\partial_t v|^2 \mathrm{d}x + \tfrac1\varepsilon \mathbf{F}(v(t)) \bigg] \mathrm{d}t$$

for mappings $v\colon \Omega_T \to \mathbb{R}^N$. To deal with the associated problem of existence, one first has to specify the function spaces in which the minimization of $\boldsymbol{\mathcal{F}}_\varepsilon$ should take place. The natural class $\boldsymbol{\mathcal{K}}$ of functions consists of those $v \in L^p\big(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) = 0$. We now consider

$$\mathcal{D}_{u_o}(\mathbf{F}) := \big(u_o + \boldsymbol{\mathcal{K}}\big) \cap \big\{\mathbf{F}(v) \in L^1(0,T)\big\},$$

which is non-empty. Indeed the time-independent extension of u_o to Ω_T belongs to this class, since

$$\boldsymbol{\mathcal{F}}_\varepsilon(u_o) = \big(1 - \mathrm{e}^{-\frac{T}{\varepsilon}}\big)\mathbf{F}(u_o) < \infty.$$

At this stage it is worth to remark, that the hypothesis $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ can be used to derive an L^2-bound for v in terms of $\partial_t v$ and u_o. Indeed, for $v \in \mathcal{D}_{u_o}(\mathbf{F})$ and $t \in (0,T]$ we have

$$\begin{aligned}\|v(t)\|^2_{L^2(\Omega,\mathbb{R}^N)} &\le 2\|v(t) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &\le 2\int_\Omega \bigg|\int_0^t \partial_t v(\tau)\mathrm{d}\tau\bigg|^2 \mathrm{d}x + 2\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

$$\leq 2t\|\partial_t v\|^2_{L^2(\Omega_T,\mathbb{R}^N)} + 2\|u_o\|_{L^2(\Omega,\mathbb{R}^N)}.$$

We integrate the preceding inequality with respect to $t \in (0,T)$ and obtain

$$\|v\|^2_{L^2(\Omega_T,\mathbb{R}^N)} \leq T^2\|\partial_t v\|^2_{L^2(\Omega_T,\mathbb{R}^N)} + 2T\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)},$$

which is the desired L^2-bound in terms of u_o and $\partial_t v$. Next, we observe that $\boldsymbol{\mathcal{F}}_\varepsilon$ is well defined on $\mathcal{D}_{u_o}(\mathbf{F})$. Moreover, with (2.2)$_1$ we can easily estimate $\boldsymbol{\mathcal{F}}_\varepsilon(v)$ from below. Indeed we have

$$\boldsymbol{\mathcal{F}}_\varepsilon(v) \geq \iint_{\Omega_T} \mathrm{e}^{-\frac{t}{\varepsilon}}\Big[\tfrac{1}{2}|\partial_t v|^2 + \tfrac{\nu}{\varepsilon}|Dv|^p\Big]\mathrm{d}x\mathrm{d}t \geq \mathrm{e}^{-T/\varepsilon}\iint_{\Omega_T}\Big[\tfrac{1}{2}|\partial_t v|^2 + \tfrac{\nu}{\varepsilon}|Dv|^p\Big]\mathrm{d}x\mathrm{d}t.$$

On the one hand this implies for any $v \in \mathcal{D}_{u_o}(\mathbf{F})$ that

$$\|v\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|\partial_t v\|^2_{L^2(\Omega,\mathbb{R}^N)} \leq 2(1+T^2)e^{T/\varepsilon}\boldsymbol{\mathcal{F}}_\varepsilon(v) + 2T\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)},$$

while on the other hand, we have

$$\begin{aligned}\|v\|_{L^p(\Omega,\mathbb{R}^N)} + \|Dv\|_{L^p(\Omega,\mathbb{R}^{N\times n})} &\leq (1+C_P)\Big[\|u_o\|_{W^{1,p}(\Omega,\mathbb{R}^N)} + \|Dv\|_{L^p(\Omega,\mathbb{R}^{N\times n})}\Big] \\ &\leq (1+C_P)\Big[\|u_o\|_{W^{1,p}(\Omega,\mathbb{R}^N)} + \Big(\tfrac{\varepsilon}{\nu}e^{T/\varepsilon}\boldsymbol{\mathcal{F}}_\varepsilon(v)\Big)^{\frac{1}{p}}\Big].\end{aligned}$$

Therefore, any $\boldsymbol{\mathcal{F}}_\varepsilon$-minimizing sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{D}_{u_o}(\mathbf{F})$ is bounded in $L^2(\Omega,\mathbb{R}^N) \cap L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$, and the associated sequence of time derivatives $\partial_t u_k$ is also bounded in $L^2(\Omega,\mathbb{R}^N)$. This implies that $(u_k)_{k\in\mathbb{N}}$ is also bounded in $W^{1,1}(\Omega_T,\mathbb{R}^N)$. This allows us to pass to a non-relabeled subsequence such that

$$\begin{cases} u_k \to u & \text{strongly in } L^1(\Omega_T,\mathbb{R}^N), \\ \partial_t u_k \rightharpoondown \partial_t u & \text{weakly in } L^2(\Omega_T,\mathbb{R}^N), \\ u_k \rightharpoondown u & \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big). \end{cases}$$

Note that $u \in u_o + \mathcal{K}$. Taking into account Lemma 3.1 we have

$$\boldsymbol{\mathcal{F}}_\varepsilon(u) \leq \liminf_{k\to\infty}\boldsymbol{\mathcal{F}}_\varepsilon(u_k),$$

proving that $u \in \mathcal{D}_{u_o}(\mathbf{F})$ is the desired minimizer of $\boldsymbol{\mathcal{F}}_\varepsilon$. In view of the convexity of $\mathbf{F}$, this minimizer is unique. We have thus proven the following existence result.

Lemma 7.2. *For any given $\varepsilon \in (0,1]$ there exists a unique $\boldsymbol{\mathcal{F}}_\varepsilon$-minimizing map u_ε in the class $\mathcal{D}_{u_o}(\mathbf{F})$.*

7.1.2. *Minimality revisited.* For fixed $\varepsilon \in (0,1]$ let u_ε denote the unique minimizer of the functional $\boldsymbol{\mathcal{F}}_\varepsilon$ in the class $\mathcal{D}_{u_o}(\mathbf{F})$. We consider mappings $\varphi \in L^p\big(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$ with time derivative $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$, initial values $\varphi(0) \in L^2(\Omega,\mathbb{R}^N)$, and such that

$$\int_0^T \mathbf{F}\big((u_\varepsilon+\varphi)(t)\big)\,\mathrm{d}t < \infty. \tag{7.1}$$

For $\delta \in (0,\mathrm{e}^{-T/\varepsilon}]$ and $(x,t) \in \Omega_T$ we define

$$w_{\varepsilon,\delta}(x,t) := u_\varepsilon(x,t) + \delta\mathrm{e}^{t/\varepsilon}\zeta(t)\varphi(x,t),$$

where $\zeta \in W^{1,\infty}(0,T)$ with $0 \leq \zeta \leq 1$. We assume either $\zeta(0) = 0$ or $\varphi(0) = 0$. Then it is easy to check that $w_{\varepsilon,\delta} \in u_o + \boldsymbol{\mathcal{K}}$ and therefore we only need to establish $\boldsymbol{\mathcal{F}}_\varepsilon(w_{\varepsilon,\delta}) < \infty$. Indeed, with $\sigma(t) := \delta\mathrm{e}^{t/\varepsilon}\zeta(t) \in [0,1]$ we can write $w_{\varepsilon,\delta}(x,t)$ as convex combination $(1-\sigma(t))u_\varepsilon(x,t) + \sigma(t)\big(\varphi(x,t)+u_\varepsilon(x,t)\big)$. Therefore, the convexity of $\mathbf{F}$ yields

$$\int_0^T \mathbf{F}\big(w_{\varepsilon,\delta}(t)\big)\,\mathrm{d}t = \int_0^T \mathbf{F}\big((1-\sigma(t))u_\varepsilon(t) + \sigma(t)\big(\varphi(t)+u_\varepsilon(t)\big)\big)\,\mathrm{d}t$$

$$
\begin{aligned}
&\le \int_0^T \Big[(1-\sigma(t))\mathbf{F}\big(u_\varepsilon(t)\big) + \sigma(t)\mathbf{F}\big((\varphi+u_\varepsilon)(t)\big)\Big]\,\mathrm{d}t \\
&\le \int_0^T \mathbf{F}\big(u_\varepsilon(t)\big)\,\mathrm{d}t + \int_0^T \mathbf{F}\big((\varphi+u_\varepsilon)(t)\big)\,\mathrm{d}t < \infty,
\end{aligned}
$$

proving the finiteness of the $\boldsymbol{\mathcal{F}}_\varepsilon$-energy.

Having arrived at this stage, we can argue as in [10, § 5.2]. First we test the minimality condition against $w_{\varepsilon,\delta}$. The resulting inequality is afterwards re-written. In this step only the convexity of the integral functional $\mathbf{F}$ is used. Here, the restriction on δ comes into the play. It ensures that $\sigma(t) := \delta \mathrm{e}^{t/\varepsilon}\zeta(t) \in [0,1]$, so that we can re-write $w_{\varepsilon,\delta}(x,t)$ as above as a convex combination. The result is then multiplied by ε/δ. In the resulting inequality we pass to the limit $\delta \downarrow 0$. As final outcome of the whole procedure we have

$$
\begin{aligned}
\int_0^T \zeta(t)\mathbf{F}(u_\varepsilon(t))\,\mathrm{d}t \le \int_0^T \zeta(t)\mathbf{F}\big(u_\varepsilon(t)+\varphi(t)\big)\,\mathrm{d}t + \iint_{\Omega_T} \zeta\partial_t u_\varepsilon\cdot\varphi\,\mathrm{d}x\mathrm{d}t \\
+\,\varepsilon \iint_{\Omega_T} \big[\zeta'\partial_t u_\varepsilon\cdot\varphi + \zeta\partial_t u_\varepsilon\cdot\partial_t\varphi\big]\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{7.2}
$$

for any $\varphi \in L^p\big(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$ with $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$ satisfying (7.1), for any $\zeta \in W^{1,\infty}(0,T)$ with $0 \le \zeta \le 1$, such that either $\zeta(0) = 0$ or $\varphi(0) = 0$. It should be emphasized again, that in the whole argument leading to (7.2) only the convexity of the integral functional $\mathbf{F}$ enters.

7.1.3. *Energy estimates, weak convergence and lower semicontinuity.* What essentially is needed to deduce suitable energy estimates, is that on the level of integral functionals the assertion $\mathbf{F}([u]_h) \le [\mathbf{F}(u)]_h$ holds true. Here it is of central importance that this can be achieved assuming only the integral convexity of $\mathbf{F}$. In the above estimate, $[u]_h$ is defined with $v_o = u_o$, while $[\mathbf{F}(u)]_h$ is defined with initial value $\mathbf{F}(u_o)$. The inequality itself can be interpreted as Jensen's inequality for the time regularization, namely that the variational integral of the time regularization of $s \mapsto u(s)$ lies below the time regularization of the variational integrals $s \mapsto \mathbf{F}(u(s))$. The assertion is proved in Lemma 3.3.

Having these ingredients at hand we can argue along the line of [10, § 5.3], to conclude that the assertions (5.6) – (5.10) from that paper hold true also in our context. More precisely, for any $\varepsilon \in (0,1]$ we have

$$
\iint_{\Omega_T} |\partial_t u_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \le \mathbf{F}(u_o),
\tag{7.3}
$$

$$
\|u_\varepsilon\|^2_{L^2(\Omega_T,\mathbb{R}^N)} \le T^2\mathbf{F}(u_o) + 2T\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}
\tag{7.4}
$$

$$
\|u_\varepsilon(t_1)-u_\varepsilon(t_2)\|_{L^2(\Omega,\mathbb{R}^N)} \le \sqrt{\mathbf{F}(u_o)}\sqrt{|t-s|} \quad \forall\, 0 \le t_1 < t_2 \le T,
\tag{7.5}
$$

$$
\int_{t_1}^{t_2} \mathbf{F}\big(u_\varepsilon(t)\big)\,\mathrm{d}t \le \big(t_2-t_1+\tfrac12\varepsilon\big)\mathbf{F}(u_o) \quad \forall\, 0 \le t_1 < t_2 \le T.
\tag{7.6}
$$

Taking into account $(2.2)_1$, the energy estimate (7.6) can be converted into an energy bound for the spatial derivatives

$$
\nu \iint_{\Omega\times(t_1,t_2)} |Du_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \le \big(t_2-t_1+\tfrac12\varepsilon\big)\mathbf{F}(u_o) \quad \forall\, 0 \le t_1 < t_2 \le T.
\tag{7.7}
$$

By (7.3), (7.4), and (7.7) (with $t_1 = 0$ and $t_2 = T$) the family $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of $\boldsymbol{\mathcal{F}}_\varepsilon$-minimizing functions is bounded in $W^{1,1}(\Omega_T,\mathbb{R}^N)$. Therefore, we infer the existence of some sequence $\varepsilon_j \downarrow 0$, which we still denote by ε, and some limit function $u \in L^p(0,T;u_o + W_0^{1,p}(\Omega_T,\mathbb{R}^N))$ with $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ such that

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } L^1(\Omega_T,\mathbb{R}^N),\\ \partial_t u_\varepsilon \rightharpoondown \partial_t u & \text{weakly in } L^2(\Omega_T,\mathbb{R}^N),\\ Du_\varepsilon \rightharpoondown Du & \text{weakly in } L^p(\Omega_T,\mathbb{R}^{N\times n}).\end{cases}$$

At this point it remains to establish that

$$\int_{t_1}^{t_2} \mathbf{F}(u(t))\,\mathrm{d}t \le \liminf_{\varepsilon\downarrow 0} \int_{t_1}^{t_2} \mathbf{F}\big(u_\varepsilon(t)\big)\,\mathrm{d}t \le (t_2 - t_2)\mathbf{F}(u_o) \tag{7.8}$$

whenever $0 \le t_1 < t_2 \le T$. This, however, is a direct consequence of Lemma 3.1. Applying (7.8) with $t_1 = 0$ and $t_2 = T$ we conclude that $\mathbf{F}(u(\cdot)) \in L^1(0,T)$. It remains to show that $u(0) = u_o$. This can be derived with the argument from [10, proof of Lemma 5.1]. Overall we proved $u \in \mathcal{D}_{u_o}(\mathbf{F})$.

7.1.4. *Passage to the limit and conclusion of the proof.* In this section we pass to the limit $\varepsilon \downarrow 0$ in the sequence of $\boldsymbol{\mathcal{F}}_\varepsilon$ minimizers u_ε on Ω_T. We consider $v \in L^p\big(0,T;u_o + W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$ with $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) \in L^2(\Omega_T,\mathbb{R}^N)$. We can also assume that $\mathbf{F}(v(\cdot)) \in L^1(0,T)$, since otherwise the variational inequality trivially holds. We define $\varphi := v - u_\varepsilon$. Observe that $\varphi \in L^p\big(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$ and that $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$ and $\varphi(0) \in L^2(\Omega_T,\mathbb{R}^N)$. Moreover, the finite energy assumption (7.1) holds. This allows us to apply (7.2) with φ and ζ_θ, with the specific choice $\zeta_\theta(t) = \frac{1}{\theta}t$ for $t \in [0,\theta)$, $\zeta_\theta(t) = 1$ for $t \in [\theta, T-\theta]$, and $\zeta_\theta(t) = \frac{1}{\theta}(T-t)$ for $t \in (T-\theta, T]$. With these choices (7.2) turns into

$$\begin{aligned}\int_0^T &\mathbf{F}\big(u_\varepsilon(t)\big)\mathrm{d}t\\ &\le \int_0^T \big(1-\zeta_\theta(t)\big)\mathbf{F}\big(u_\varepsilon(t)\big)\mathrm{d}t + \iint_{\Omega_T} \zeta_\theta \partial_t u_\varepsilon \cdot (v-u_\varepsilon)\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^T \mathbf{F}(v(t))\mathrm{d}t + \varepsilon \iint_{\Omega_T} \Big[\zeta_\theta' \partial_t u_\varepsilon \cdot (v-u_\varepsilon) + \zeta_\theta \partial_t u_\varepsilon \cdot \partial_t(v-u_\varepsilon)\Big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The integrals on the right can now be treated exactly as in [10, § 5.4]. We first pass to the limit $\varepsilon \downarrow 0$. Here, we apply the lower-semicontinuity from (7.8) to the left-hand side integral, while for the initial boundary term we use (7.5). After that we pass to the limit $\theta \downarrow 0$. This yields the variational inequality for the weak limit u, i.e. we have

$$\begin{aligned}\int_0^T \mathbf{F}(u(t))\mathrm{d}t \le \int_0^T \mathbf{F}(v(t))\mathrm{d}t + \iint_{\Omega_T} \partial_t u \cdot (v-u)\mathrm{d}x\mathrm{d}t&\\ + \tfrac{1}{2}\big\|v(0)-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)} - \tfrac{1}{2}\big\|(v-u)(T)\big\|^2_{L^2(\Omega,\mathbb{R}^N)}&,\end{aligned}$$

for any $v \in L^p\big(0,T;W_{u_o}^{1,p}(\Omega,\mathbb{R}^N)\big)$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) \in L^2(\Omega_T,\mathbb{R}^N)$, such that $\mathbf{F}(v(\cdot)) \in L^1(0,T)$. This completes the proof of Theorem 7.1.

7.2. **Proof via Minimizing Movements.** In this subsection we prove the sufficiency part of Theorem 2.3 in the general case.

Theorem 7.3. *Assume that $p > 1$ and that u_* and $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ satisfy (2.1) and (2.2) and that $\mathbf{F}$ is convex on $L^2(\Omega,\mathbb{R}^N) \cap u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)$. Then, for any $u_o \in L^2(\Omega,\mathbb{R}^N)$ there exists a variational solution $u \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;u_* + W_0^{1,p}(\Omega,\mathbb{R}^N))$ to the gradient flow for $\mathbf{F}$ in the sense of Definition 2.1.*

We redefine the space $\mathcal{D}_{u_*}(\mathbf{F})$ as

$$\mathcal{D}_{u_*}(\mathbf{F}) := \big\{u \in u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)\colon \mathbf{F}(u) < \infty\big\}.$$

7.2.1. *Time discretization.* For $k \in \mathbb{N}$, we consider the step size $h_k := \frac{T}{k}$ and associated times $t_{k,i} := ih_k$ for $i \in \{-1, \dots, k\}$. The strategy is to construct approximations $u^{(k)} \colon \Omega \times (-h_k, T] \to \mathbb{R}^N$ that are constant in time on each of the time intervals $(t_{k,i-1}, t_{k,i}]$, i.e.

$$u^{(k)}(t) := u_{k,i} \quad \text{for } t \in (t_{k,i-1}, t_{k,i}] \text{ with } i \in \{0, \dots, k\}.$$

The approximations $u_{k,i}$ on the separate time intervals are determined by solving an elliptic minimization problem. We begin by defining $u_{k,0} := u_o$. Then, assuming that the map $u_{k,i-1} \in L^2(\Omega, \mathbb{R}^N)$ has already been defined for some $i \in \{1, \dots, k\}$, we define $u_{k,i} \in \mathcal{D}_{u_*}(\mathbf{F}) \cap L^2(\Omega, \mathbb{R}^N)$ as minimizer of the convex functional

$$\mathbf{F}_{k,i}(v) := \mathbf{F}(v) + \tfrac{1}{2h_k} \int_\Omega |v - u_{k,i-1}|^2 \mathrm{d}x$$

in the class $\mathcal{D}_{u_*}(\mathbf{F}) \cap L^2(\Omega, \mathbb{R}^N)$. The existence of this minimizer follows from the Direct Method of the Calculus of Variation due to the convexity of the functional $\mathbf{F}$. Indeed, from the coercivity assumption $(2.2)_1$ and Poincaré's inequality we have

$$\begin{aligned}\|v\|_{L^p(\Omega,\mathbb{R}^N)} &\le \|u_*\|_{L^p(\Omega,\mathbb{R}^N)} + C_P\|Dv - Du_*\|_{L^p(\Omega,\mathbb{R}^{N\times n})} \\ &\le \|u_*\|_{L^p(\Omega,\mathbb{R}^N)} + C_P\|Du_*\|_{L^p(\Omega,\mathbb{R}^{N\times n})} + C_P\|Dv\|_{L^p(\Omega,\mathbb{R}^{N\times n})}.\end{aligned}$$

Therefore, an $\mathbf{F}_{k,i}$-minimizing sequence $(v_\ell)_{\ell\in\mathbb{N}}$ of maps $v_\ell \in \mathcal{D}_{u_*}(\mathbf{F}) \cap L^2(\Omega_T, \mathbb{R}^N)$ is uniformly bounded in $L^2(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$, and we can pass to (a non-relabeled subsequence) such that $v_\ell \rightharpoonup u_{k,i}$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$ and weakly in $L^2(\Omega, \mathbb{R}^N)$ for some function $u_{k,i} \in L^2(\Omega, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N)$. Due to the lower semi-continuity of $\mathbf{F}$ with respect to weak convergence we arrive at

$$\mathbf{F}_{k,i}(u_{k,i}) \le \liminf_{\ell\to\infty} \mathbf{F}_{k,i}(v_\ell) = \inf_{v \in D_{u_*}(\mathbf{F})\cap L^2(\Omega,\mathbb{R}^N)} \mathbf{F}_{k,i}(v). \tag{7.9}$$

Note that the minimizer is unique.

7.2.2. *Minimizing property of the approximations.* We let $k \in \mathbb{N}$. For $\tau \in (0, T]$ we define the functional

$$\mathbf{F}^{(k)}(v) := \int_0^\tau \mathbf{F}(v(t))\,\mathrm{d}t + \tfrac{1}{2h_k} \iint_{\Omega_\tau} \big|v(t) - u^{(k)}(t - h_k)\big|^2 \mathrm{d}x\mathrm{d}t$$

for functions $v \in L^2(\Omega_T, \mathbb{R}^N) \cap L^p(0, T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$. For $j := \lceil \frac{\tau}{h_k} \rceil$ we have

$$\begin{aligned}\mathbf{F}^{(k)}\big(u^{(k)}\big) &= \sum_{i=1}^{j} \int_{(i-1)h_k}^{\tau\wedge ih_k} \Big[\mathbf{F}(u_{k,i}) + \tfrac{1}{2h_k}\int_\Omega |u_{k,i} - u_{k,i-1}|^2 \mathrm{d}x\Big]\mathrm{d}t \\ &= \sum_{i=1}^{j} \int_{(i-1)h_k}^{\tau\wedge ih_k} \mathbf{F}_{k,i}(u_{k,i})\mathrm{d}t \\ &\le \sum_{i=1}^{j} \int_{(i-1)h_k}^{\tau\wedge ih_k} \mathbf{F}_{k,i}(v(t))\mathrm{d}t \\ &= \sum_{i=1}^{j} \int_{(i-1)h_k}^{\tau\wedge ih_k} \Big[\mathbf{F}(v(t)) + \tfrac{1}{2h_k}\int_\Omega \big|v(t) - u^{(k)}(t - h_k)\big|^2 \mathrm{d}x\Big]\mathrm{d}t \\ &= \mathbf{F}^{(k)}(v),\end{aligned}$$

for any function v as above. This shows that $u^{(k)}$ minimizes the functional $\mathbf{F}^{(k)}$ in the function class $L^2(\Omega_T, \mathbb{R}^N) \cap L^p(0, T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$. For any $s \in (0, 1)$ and $v \in L^2(\Omega_T, \mathbb{R}^N) \cap L^p(0, T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$ we consider the convex combination of $u^{(k)}$ and v defined by

$$w := sv + (1 - s)u^{(k)} \in L^2(\Omega_T, \mathbb{R}^N) \cap L^p\big(0, T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N)\big).$$

Using the minimality property of $u^{(k)}$ shown above and the convexity of $\mathbf{F}$, we obtain

$$\begin{aligned}
&\int_0^\tau \mathbf{F}\big(u^{(k)}\big)\mathrm{d}t\\
&\quad\le \int_0^\tau \mathbf{F}(w)\,\mathrm{d}t\\
&\qquad+\tfrac{1}{2h_k}\iint_{\Omega_\tau}\Big[\big|w-u^{(k)}(t-h_k)\big|^2-\big|u^{(k)}-u^{(k)}(t-h_k)\big|^2\Big]\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_0^\tau \Big[s\mathbf{F}(v)+(1-s)\mathbf{F}\big(u^{(k)}\big)\Big]\,\mathrm{d}t\\
&\qquad+\tfrac{1}{h_k}\iint_{\Omega_\tau}\Big[\tfrac{s^2}{2}\big|v-u^{(k)}\big|^2+s\big(v-u^{(k)}\big)\cdot\big(u^{(k)}-u^{(k)}(t-h_k)\big)\Big]\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for any $v\in L^2(\Omega_T,\mathbb{R}^N)\cap L^p(0,T;u_*+W_0^{1,p}(\Omega,\mathbb{R}^N))$. After re-absorbing the second integral appearing on the right-hand side on the left, dividing the result by $s>0$ and letting $s\downarrow 0$, we find

$$\int_0^\tau \mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t\le \int_0^\tau \mathbf{F}(v)\,\mathrm{d}t+\tfrac{1}{h_k}\iint_{\Omega_\tau}\big(v-u^{(k)}\big)\cdot\big(u^{(k)}-u^{(k)}(t-h_k)\big)\,\mathrm{d}x\mathrm{d}t.$$

We extend v to negative times $t<0$ by letting $v(t):=v(0)\in L^2(\Omega,\mathbb{R}^N)$ and continue the computation as follows:

$$\begin{aligned}
\int_0^\tau \mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t &\le \int_0^\tau \mathbf{F}(v)\,\mathrm{d}t+\tfrac{1}{h_k}\iint_{\Omega_\tau}\big(v-u^{(k)}\big)\cdot\big(v-v(t-h_k)\big)\mathrm{d}x\mathrm{d}t\\
&\quad+\tfrac{1}{2h_k}\iint_{\Omega_\tau}\Big[\big|v-u^{(k)}\big|^2(t-h_k)-\big|v-u^{(k)}\big|^2\Big]\mathrm{d}x\mathrm{d}t\\
&\quad-\tfrac{1}{2h_k}\iint_{\Omega_\tau}\big|v-v(t-h_k)-u^{(k)}+u^{(k)}(t-h_k)\big|^2\mathrm{d}x\mathrm{d}t\\
&\le \int_0^\tau \mathbf{F}(v)\,\mathrm{d}t+\tfrac{1}{h_k}\iint_{\Omega_\tau}\big(v-u^{(k)}\big)\cdot\big(v-v(t-h_k)\big)\mathrm{d}x\mathrm{d}t\\
&\quad-\tfrac{1}{2h_k}\iint_{\Omega\times[\tau-h_k,\tau]}\big|v-u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t+\tfrac12\int_\Omega|v(0)-u_o|^2\mathrm{d}x\,,
\end{aligned}\tag{7.10}$$

where in the last line we used the fact that $v(t)=v(0)$ for $t\le 0$. This inequality holds for any $\tau\in(0,T]$ and any $v\in L^2(\Omega_T,\mathbb{R}^N)\cap L^p(0,T;u_*+W_0^{1,p}(\Omega,\mathbb{R}^N))$.

7.2.3. *Energy estimates.* Our next goal is to derive energy estimates for the sequence of approximations $u^{(k)}$, which will allow us to pass to the limit $k\to\infty$. In (7.10) we choose $v=u_*$ and $\tau=jh_k$ with $j\in\{1,\dots,k\}$. This yields

$$\int_0^{jh_k}\mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t+\tfrac12\int_\Omega\big|u^{(k)}(jh_k)-u_*\big|^2\mathrm{d}x\le T\,\mathbf{F}(u_*)+\tfrac12\int_\Omega|u_*-u_o|^2\mathrm{d}x,$$

and in turn we obtain

$$\int_0^{jh_k}\mathbf{F}\big(u^{(k)}\big)\mathrm{d}t+\tfrac14\int_\Omega\big|u^{(k)}(jh_k)\big|^2\,\mathrm{d}x\le M,\tag{7.11}$$

where

$$M:=T\,\mathbf{F}(u_*)+\int_\Omega\big[|u_o|^2+\tfrac32|u_*|^2\big]\,\mathrm{d}x.$$

This shows

$$\sup_{t\in[0,T]}\int_\Omega\big|u^{(k)}(t)\big|^2\,\mathrm{d}x\le 4M,$$

and by assumption $(2.2)_1$ also

$$\iint_{\Omega_T} \big|Du^{(k)}\big|^p \,\mathrm{d}x\mathrm{d}t \le \tfrac{1}{\nu} \int_0^T \mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t \le \tfrac{1}{\nu} M. \tag{7.12}$$

Note that due to Poincaré's inequality we also have

$$\begin{aligned}\big\|u^{(k)}\big\|_{L^p(\Omega_T,\mathbb{R}^N)} &\le T\|u_*\|_{L^p(\Omega,\mathbb{R}^N)} + C_P\big\|D(u^{(k)}-u_*)\big\|_{L^p(\Omega_T,\mathbb{R}^{N\times n})} \\ &\le C_P\Big[T\|u_*\|_{W^{1,p}(\Omega,\mathbb{R}^N)} + \big(\tfrac{1}{\nu}M\big)^{\frac{1}{p}}\Big].\end{aligned}$$

We therefore know that $u^{(k)}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ and in $L^p(0,T;u_*+W^{1,p}_0(\Omega,\mathbb{R}^N))$.

Since we do not impose that u_o has finite energy, we cannot expect to obtain uniform estimates for the difference quotient in time on the whole space-time cylinder Ω_T. However, solutions become more regular immediately after the initial time $t=0$. In fact, in the following we obtain improved energy estimates on cylinders $\Omega\times(\varepsilon,T]$ with $\varepsilon>0$. The argument is as follows. For $i\in\{2,\dots,k\}$, we consider the functional $\mathbf{F}_{k,i}$. From the minimizing property of $u_{k,i}$ in (7.9) we obtain with the choice of $\varphi=u_{k,i-1}\in\mathcal{D}_{u_*}(\mathbf{F})\cap L^2(\Omega,\mathbb{R}^N)$ as comparison function that

$$\mathbf{F}(u_{k,i}) + \tfrac{1}{2h_k}\int_\Omega |u_{k,i}-u_{k,i-1}|^2\mathrm{d}x \le \mathbf{F}(u_{k,i-1}).$$

Note that the choice $i=1$ is not allowed since we do not know that u_o has finite energy. We sum this inequality from $i=j_1+1,\dots,j_2$ with $1\le j_1<j_2\le k$. This yields

$$\sum_{i=j_1+1}^{j_2}\bigg[\mathbf{F}(u_{k,i}) + \tfrac{1}{2h_k}\int_\Omega |u_{k,i}-u_{k,i-1}|^2\mathrm{d}x\bigg] \le \sum_{i=j_1}^{j_2-1}\mathbf{F}(u_{k,i}),$$

which implies

$$\mathbf{F}(u_{k,j_2}) + \tfrac{1}{2h_k}\sum_{i=j_1+1}^{j_2}\int_\Omega |u_{k,i}-u_{k,i-1}|^2\mathrm{d}x \le \mathbf{F}(u_{k,j_1}) \tag{7.13}$$

for any $1\le j_1<j_2\le k$. Since all terms are non-negative this implies in particular that $\{1,\dots,k\}\ni j\mapsto \mathbf{F}(u_{k,j})$ is decreasing and hence

$$\mathbf{F}(u_{k,j_2}) + \tfrac{1}{2h_k}\sum_{i=j_1+1}^{j_2}\int_\Omega |u_{k,i}-u_{k,i-1}|^2\mathrm{d}x \le \mathbf{F}(u_{k,j})$$

for any $j\in\{1,\dots,j_1\}$. Recalling the definition of $u^{(k)}$, this inequality can be re-written as

$$\mathbf{F}\big(u^{(k)}(j_2h_k)\big) + \tfrac12\iint_{\Omega\times(j_1h_k,j_2h_k)}\big|\Delta_{-h_k}u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t \le \mathbf{F}\big(u^{(k)}(t)\big),$$

for any $t\in[h_k,j_1h_k]$, where Δ_{-h_k} denotes the backwards difference quotient in time. We now let $0<\varepsilon<\tau\le T$. Then, for $k\in\mathbb{N}$ with $k>\frac{4T}{\varepsilon}$, $j_1=\lfloor\frac{\varepsilon}{h_k}\rfloor$ and $j_2=\lceil\frac{\tau}{h_k}\rceil$, we obtain

$$\mathbf{F}\big(u^{(k)}(\tau)\big) + \tfrac12\iint_{\Omega\times(\varepsilon,\tau)}\big|\Delta_{-h_k}u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t \le \mathbf{F}\big(u^{(k)}(t)\big),$$

for any $t\in[h_k,\varepsilon-h_k]$. Observe that the left-hand side is independent of t. Taking mean values with respect to $t\in[h_k,\varepsilon-h_k]$ and using (7.12) therefore yields

$$\mathbf{F}\big(u^{(k)}(\tau)\big) + \tfrac12\iint_{\Omega\times(\varepsilon,\tau)}\big|\Delta_{-h_k}u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t \le \fint_{h_k}^{\varepsilon-h_k}\mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t \le \frac{M}{\varepsilon-2h_k}, \tag{7.14}$$

for any $\tau\in(\varepsilon,T]$.

7.2.4. *The limit map.* The energy bounds from the last section imply that the sequence $(u^{(k)})_{k\in\mathbb{N}}$ is uniformly bounded in the spaces $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ and $L^p(0,T;u_*+W^{1,p}_0(\Omega,\mathbb{R}^N))$. Therefore, there exists a limit map

$$u\in L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big)\cap L^p\big(0,T;u_*+W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$$

and a subsequence $\mathfrak{K}\subset\mathbb{N}$ such that

$$\left\{\begin{array}{ll} u^{(k)}\overset{*}{\rightharpoonup} u & \text{weakly}^* \text{ in } L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big),\\ u^{(k)}\rightharpoonup u & \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big),\end{array}\right. \tag{7.15}$$

as $\mathfrak{K}\ni k\to\infty$. Next, we define an auxiliary function $\tilde u^{(k)}\colon\Omega\times(-h_k,T]\to\mathbb{R}^N$ as the linear interpolation of $u_{k,i-1}$ and $u_{k,i}$ on the interval $((i-1)h_k,ih_k]$. The precise definition is

$$\tilde u^{(k)}(t):=\big(i-\tfrac{t}{h_k}\big)u_{k,i-1}+\big(1-i+\tfrac{t}{h_k}\big)u_{k,i} \text{ for } t\in((i-1)h_k,ih_k] \text{ with } i\in\{1,\dots,k\},$$

and $\tilde u^{(k)}(t):=u_o$ for $t\in(-h_k,0]$. For $t\in((i-1)h_k,ih_k]$ we compute

$$\partial_t\tilde u^{(k)}=\tfrac{1}{h_k}\big(u_{k,i}-u_{k,i-1}\big)\in L^2(\Omega,\mathbb{R}^N)$$

which due to (7.14) and hypothesis $(2.2)_1$ yields

$$\nu\sup_{t\in[\varepsilon,T]}\big\|D\tilde u^{(k)}(t)\big\|^p_{L^p(\Omega,\mathbb{R}^{N\times n})}+\tfrac12\iint_{\Omega\times(\varepsilon,T)}\big|\partial_t\tilde u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t\le\frac{M}{\varepsilon-2h_k}, \tag{7.16}$$

for any $\varepsilon\in(0,T)$. We now fix $\varepsilon\in(0,T)$ and note that $\tilde u^{(k)}$ satisfies the same energy bounds as $u^{(k)}$ in $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ and $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$. Therefore, there exists a limit map

$$\tilde u\in L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big)\cap L^p\big(0,T;u_*+W^{1,p}_0(\Omega,\mathbb{R}^N)\big)$$

with

$$\tilde u\in L^\infty\big(\varepsilon,T;u_*+W^{1,p}_0(\Omega,\mathbb{R}^N)\big) \text{ and } \partial_t\tilde u\in L^2\big(\Omega\times(\varepsilon,T],\mathbb{R}^N\big),$$

and a subsequence $\mathfrak{K}_1\subset\mathfrak{K}$ such that

$$\left\{\begin{array}{ll} \tilde u^{(k)}\rightharpoonup\tilde u & \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big),\\ \tilde u^{(k)}\overset{*}{\rightharpoonup}\tilde u & \text{weakly}^* \text{ in } L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big),\\ \tilde u^{(k)}\to\tilde u & \text{strongly in } L^{\min\{2,p\}}(\Omega\times(\varepsilon,T],\mathbb{R}^N),\\ \partial_t\tilde u^{(k)}\rightharpoonup\partial_t\tilde u & \text{weakly in } L^2(\Omega\times(\varepsilon,T],\mathbb{R}^N),\end{array}\right. \tag{7.17}$$

in the limit $\mathfrak{K}_1\ni k\to\infty$. Since

$$\big|\big(\tilde u^{(k)}-u^{(k)}\big)(t)\big|\le|u_{k,i}-u_{k,i-1}|\quad\text{for } t\in((i-1)h_k,ih_k],$$

we conclude from (7.13) and (7.11) for $k\in\mathbb{N}$ with $k>\frac{T}{\varepsilon}$ (which implies $h_k<\varepsilon$) that

$$\iint_{\Omega\times(\varepsilon,T)}\big|\tilde u^{(k)}-u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t\le h_k\sum_{i=2}^k\int_\Omega|u_{k,i}-u_{k,i-1}|^2\mathrm{d}x\le2h_k^2\,\mathbf{F}(u_{k,1})\le2h_kM.$$

By Hölder's inequality this implies

$$\iint_{\Omega\times(\varepsilon,T)}\big|\tilde u^{(k)}-u^{(k)}\big|\mathrm{d}x\mathrm{d}t\le\sqrt{2h_kM|\Omega_T|},$$

so that together with $(7.17)_3$ we conclude that also $u^{(k)}\to\tilde u$ strongly in $L^1(\Omega\times(\varepsilon,T])$ as $\mathfrak{K}_1\ni k\to\infty$. Since $\varepsilon\in(0,T)$ was arbitrary, this implies $\tilde u=u$ and hence we have that $\partial_tu\in L^2(\Omega\times(\varepsilon,T],\mathbb{R}^N)$ for any $\varepsilon\in(0,T)$. Moreover, (7.16) yields the bound

$$\nu\sup_{t\in[\varepsilon,T]}\|Du(t)\|^p_{L^p(\Omega,\mathbb{R}^{N\times n})}+\tfrac12\iint_{\Omega\times(\varepsilon,T)}|\partial_tu|^2\mathrm{d}x\mathrm{d}t\le\frac{M}{\varepsilon}.$$

7.2.5. *Variational inequality for the limit map.* Our aim here is to pass to the limit $k \to \infty$ in (7.10). To this aim we consider

$$v \in L^p\big(0,T;u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)\big) \text{ with } \partial_t v \in L^2(\Omega_T,\mathbb{R}^N) \text{ and } v(0) \in L^2(\Omega,\mathbb{R}^N).$$

We extend v to negative times $t<0$ by $v(t) := v(0) \in L^2(\Omega,\mathbb{R}^N)$ and integrate inequality (7.10) with respect to τ over $(t_o, t_o+\delta) \subset [0,T]$ and divide the result by δ. By Fubini's theorem we therefore obtain

$$\begin{aligned}\int_0^{t_o} \mathbf{F}\big(u^{(k)}\big)\,\mathrm{d}t + \tfrac{1}{2\delta}\iint_{\Omega\times(t_o-h_k,t_o+\delta)} \big|v-u^{(k)}\big|^2\mathrm{d}x\mathrm{d}t \\ \le \fint_{t_o}^{t_o+\delta}\bigg[\int_0^\tau \mathbf{F}(v)\,\mathrm{d}t + \iint_{\Omega_\tau} \Delta_{-h_k}v\cdot\big(v-u^{(k)}\big)\,\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\tau \\ + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}.\end{aligned}$$

On the right-hand side we can pass to the limit $k \to \infty$ due to the weak convergence $u^{(k)} \rightharpoonup u$ in $L^2(\Omega_T,\mathbb{R}^N)$ and the fact that $\Delta_{-h_k}v \to \partial_t v$ strongly in $L^2(\Omega_T,\mathbb{R}^N)$. For the first term on the left-hand side we apply Lemma 3.1 and Remark 3.2 and finally for the second term we use the lower semi-continuity with respect to weak convergence of $u^{(k)} \rightharpoonup u$ in $L^2(\Omega_T,\mathbb{R}^N)$. Therefore, we obtain in the limit $\mathfrak{K}_1 \ni k \to \infty$ that

$$\begin{aligned}\int_0^{t_o} \mathbf{F}(u)\,\mathrm{d}t + \tfrac{1}{2\delta}\iint_{\Omega\times(t_o,t_o+\delta)} |v-u|^2\mathrm{d}x\mathrm{d}t \\ \le \fint_{t_o}^{t_o+\delta}\bigg[\int_0^\tau \mathbf{F}(v)\,\mathrm{d}t + \iint_{\Omega_\tau} \partial_t v\cdot(v-u)\,\mathrm{d}x\mathrm{d}t\bigg]\mathrm{d}\tau + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}.\end{aligned}$$

Passing to the limit $\delta \downarrow 0$ this yields the variational inequality

$$\begin{aligned}\int_0^{t_o} \mathbf{F}(u)\,\mathrm{d}t - \tfrac12\|(v-u)(t_o)\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ \le \int_0^{t_o} \mathbf{F}(v)\,\mathrm{d}t + \iint_{\Omega_{t_o}} \partial_t v\cdot(v-u)\,\mathrm{d}x\mathrm{d}t + \tfrac12\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}\end{aligned}$$

for any $t_o \in (0,T]$ and any $v \in L^p\big(0,T;u_*+W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) \in L^2(\Omega,\mathbb{R}^N)$. This shows that u satisfies the variational inequality (2.3).

7.2.6. *Continuity in time.* Since $\partial_t u \in L^2(\Omega\times(\varepsilon,T],\mathbb{R}^N)$ for any $\varepsilon\in(0,T)$ we already know that $u \in C^0((0,T];L^2(\Omega,\mathbb{R}^N))$. Therefore, it remains to establish continuity in $t=0$. For $\varepsilon>0$ we consider the inner parallel set Ω^ε of Ω and let $\phi \in C_0^\infty(B_1(0),\mathbb{R}_{\ge 0})$ be a standard mollifier. We set $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$, so that $\phi_\varepsilon \in C_0^\infty(B_\varepsilon(0),\mathbb{R}_{\ge0})$. We define the mollification of the initial values u_o by

$$u_o^{(\varepsilon)} := u_* + \big((u_o-u_*)\chi_{\Omega^{2\varepsilon}}\big) * \phi_\varepsilon,$$

where $\chi_{\Omega^{2\varepsilon}}$ denotes the characteristic function of $\Omega^{2\varepsilon}$. Then, we have $u_o^{(\varepsilon)} \in u_* + C_0^\infty(\Omega,\mathbb{R}^N)$ and therefore, assumption $(2.2)_3$ implies $\mathbf{F}(u_o^{(\varepsilon)}) < \infty$, so that Lemma 3.3 is applicable with $v_o = u_o^{(\varepsilon)}$. Moreover, we have $u_o^{(\varepsilon)} \to u_o \in L^2(\Omega,\mathbb{R}^N)$. We test the variational inequality (2.3) with $v = [u]_{\lambda,\varepsilon}$, where $[u]_{\lambda,\varepsilon}$ denotes the mollification with respect to time from (3.2) with the choice $v_o \equiv u_o^{(\varepsilon)}$ as initial value. This leads to

$$\begin{aligned}&\tfrac12\big\|\big([u]_{\lambda,\varepsilon}-u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \mathbf{F}(u)\,\mathrm{d}t \\ &\le \tfrac12\big\|u_o^{(\varepsilon)}-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \mathbf{F}\big([u]_{\lambda,\varepsilon}\big)\,\mathrm{d}t + \iint_{\Omega_\tau} \partial_t[u]_{\lambda,\varepsilon}\cdot\big([u]_{\lambda,\varepsilon}-u\big)\mathrm{d}x\mathrm{d}t \\ &\le \tfrac12\big\|u_o^{(\varepsilon)}-u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \big[\mathbf{F}(u)\big]_{\lambda,\varepsilon}\mathrm{d}t,\end{aligned}$$

for any $\tau \in (0,T]$. In the last line we used Lemma 3.3 and identity (3.3) for the time mollification which allowed us to discard the negative last integral. Note that $[\mathbf{F}(u)]_{\lambda,\varepsilon}$ is defined according to (3.2) with the choice $v_o = \mathbf{F}(u_o^{(\varepsilon)})$. In the first term appearing on the left-hand side we now replace $[u]_{\lambda,\varepsilon}$ by $[u]_\lambda$, where, according to (3.2), $[u]_\lambda$ is defined with initial values $v_o = u_o$. This leads to

$$\begin{aligned}\big\|\big([u]_\lambda - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} &\le 4\int_0^\tau \Big[\big[\mathbf{F}(u)\big]_{\lambda,\varepsilon} - \mathbf{F}(u)\Big]\,\mathrm{d}t + 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &= -4\lambda\int_0^\tau \partial_t\big[\mathbf{F}(u)\big]_{\lambda,\varepsilon}\,\mathrm{d}t + 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}\\ &\le 4\lambda\,\mathbf{F}\big(u_o^{(\varepsilon)}\big) + 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)},\end{aligned}$$

for any $\tau \in (0,T]$. In the second last line we used again identity (3.3). Now, we consider a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i \downarrow 0$ and choose

$$\lambda_i := \min\Big\{\varepsilon_i, \mathbf{F}\big(u_o^{(\varepsilon_i)}\big)^{-2}\Big\},$$

so that also $\lambda_i \downarrow 0$ as $i \to \infty$. Using the preceding inequality with λ_i, we obtain for any $i \in \mathbb{N}$ that

$$\sup_{\tau\in(0,T]}\big\|\big([u]_{\lambda_i} - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 4\sqrt{\lambda_i} + 4\big\|u_o^{(\varepsilon_i)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)}.$$

Since the left-hand side converges to zero in the limit $i \to \infty$, we conclude that

$$\lim_{i\to\infty}\sup_{\tau\in(0,T]}\big\|\big([u]_{\lambda_i} - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} = 0.$$

Keeping in mind that $[u]_{\lambda_i} \in C^0\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big)$ with $[u]_{\lambda_i}(0) = u_o$ for any $i \in \mathbb{N}$, we deduce from the above convergence that also $u \in C^0\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big)$ is true and that $u(0) = u_o$. This proves the desired continuity property with respect to time. We have thus proved that u is a variational solution of the gradient flow associated to $\mathbf{F}$ in the sense of Definition 2.1 satisfying additionally $\partial_t u \in L^2(\Omega \times (\varepsilon,T],\mathbb{R}^N)$ for any $\varepsilon \in (0,T)$. This finishes the proof of Theorem 7.3. □

7.3. **Uniqueness.** Here we prove that variational solutions are unique if $\mathbf{F}$ is convex. We emphasize that we do not need to impose its strict convexity.

Theorem 7.4. *Assume that $p > 1$ and that $\mathbf{F}\colon W^{1,p}(\Omega,\mathbb{R}^N) \to [0,\infty]$ is convex and u_* satisfies (2.1). Then, there exists at most one variational solution to the gradient flow associated to $\mathbf{F}$ with initial datum $u_o \in L^2(\Omega,\mathbb{R}^N)$ in the sense of Definition 2.1 if one of the following conditions is satisfied:*

(i) $u_o = u_*$,

(ii) $\mathbf{F}$ *is finite on* $u_* + C_0^\infty(\Omega,\mathbb{R}^N)$.

Proof. We suppose that

$$u_1, u_2 \in C^0\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big) \cap L^p\big(0,T; u_* + W_0^{1,p}(\Omega,\mathbb{R}^N)\big)$$

are two variational solutions in the sense of Definition 2.1. In the following we abbreviate $w := \frac{u_1+u_2}{2}$. Adding the variational inequalities (2.3) for u_1 and u_2 yields

$$\begin{aligned}&\tfrac12\|(v-u_1)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12\|(v-u_2)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \big[\mathbf{F}(u_1) + \mathbf{F}(u_2)\big]\,\mathrm{d}t\\ &\qquad \le \|v(0) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2\int_0^\tau \mathbf{F}(v)\,\mathrm{d}t + 2\iint_{\Omega_\tau} \partial_t v\cdot(v-w)\,\mathrm{d}x\mathrm{d}t \qquad (7.18)\end{aligned}$$

for any $\tau \in (0,T]$ and any $v \in L^p(0,T; u_* + W_0^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and $v(0) \in L^2(\Omega,\mathbb{R}^N)$.

We first prove the uniqueness in the simpler setting (i) when $u_o = u_*$. In this case we know from Lemma 3.4 that $\partial_t u_1, \partial_t u_2 \in L^2(\Omega_T, \mathbb{R}^N)$. With the choice $v = w$ and the convexity of $\mathbf{F}$ we find that

$$\tfrac14 \|(u_1 - u_2)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \big[\mathbf{F}(u_1) + \mathbf{F}(u_2)\big]\,\mathrm{d}t \le 2\int_0^\tau \mathbf{F}(w)\,\mathrm{d}t \le \int_0^\tau \big[\mathbf{F}(u_1) + \mathbf{F}(u_2)\big]\,\mathrm{d}t$$

for any $\tau \in (0,T]$. This shows that $u_1 = u_2$.

If assumption (ii) is satisfied we know that $\partial_t u_1, \partial_t u_2 \in L^2(\Omega \times (\delta, T], \mathbb{R}^N)$ only for any $\delta \in (0,T)$. Therefore, we have to use a mollification argument. For $\varepsilon > 0$ we consider the inner parallel set Ω^ε of Ω and let $\phi \in C_0^\infty(B_1(0), \mathbb{R}_{\ge 0})$ be a standard mollifier. We set $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$, so that $\phi_\varepsilon \in C_0^\infty(B_\varepsilon(0), \mathbb{R}_{\ge 0})$. We define the mollification of the initial values by

$$u_o^{(\varepsilon)} := u_* + \big((u_o - u_*)\chi_{\Omega^{2\varepsilon}}\big) * \phi_\varepsilon,$$

where $\chi_{\Omega^{2\varepsilon}}$ denotes the characteristic function of $\Omega^{2\varepsilon}$. Then, $u_o^{(\varepsilon)} \in u_* + C_0^\infty(\Omega, \mathbb{R}^N)$ and therefore $\mathbf{F}(u_o^{(\varepsilon)}) < \infty$ by assumption (ii), so that Lemma 3.3 is applicable with $v_o = u_o^{(\varepsilon)}$. Moreover, $u_o^{(\varepsilon)} \to u_o \in L^2(\Omega, \mathbb{R}^N)$ as $\varepsilon \downarrow 0$. By $[u_i]_h$ for $i \in \{1,2\}$ with $h \in (0,T]$ we denote the time mollification of u_i as in (3.2) with initial value $v_o \equiv u_o^{(\varepsilon)}$. Note that $[u_i]_h \in L^p(0,T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$ and $\partial_t [u_i]_h \in L^2(\Omega_T, \mathbb{R}^N)$, cf. [8, Appendix B]. We now choose $v = \frac12\big([u_1]_h + [u_2]_h\big)$ as comparison map in (7.18). Using (3.3) we obtain

$$2\partial_t v \cdot (v - w) = \partial_t v \cdot (v - u_1) + \partial_t v \cdot (v - u_2) = -\tfrac{1}{2h}\big|[u_1]_h - u_1 + [u_2]_h - u_2\big|^2 \le 0$$

and therefore

$$\begin{aligned}\tfrac12 \|(v - u_1)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \tfrac12 \|(v - u_2)(\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \big[\mathbf{F}(u_1) + \mathbf{F}(u_2)\big]\,\mathrm{d}t \\ \le \big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2\int_0^\tau \mathbf{F}(v)\,\mathrm{d}t \\ \le \big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)} + \int_0^\tau \big[\mathbf{F}\big([u_1]_h\big) + \mathbf{F}\big([u_2]_h\big)\big]\,\mathrm{d}t.\end{aligned}$$

From Lemma 3.3 we know that

$$\lim_{h\downarrow 0} \int_0^\tau \mathbf{F}\big([u_i]_h\big)\,\mathrm{d}t = \int_0^\tau \mathbf{F}(u_i)\,\mathrm{d}t.$$

Therefore, in the limit $h \downarrow 0$ the last inequality simplifies to

$$\big\|(u_1 - u_2)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 4\big\|u_o^{(\varepsilon)} - u_o\big\|^2_{L^2(\Omega,\mathbb{R}^N)},$$

for any $\tau \in (0,T]$. In the preceding inequality we let $\varepsilon \downarrow 0$. This proves the desired claim, that $u_1 = u_2$ a.e. on Ω_T.

□

7.4. **Existence of weak solutions.** The passage from the minimality condition (2.3) to a weak solution of the associated parabolic system

$$\partial_t u - \operatorname{div}\big(D_\xi f(x,u,Du)\big) + D_u f(x,u,Du) = 0 \tag{7.19}$$

is possible under certain additional assumptions on the integrand f. We assume that $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is a Carathéodory-function that for a.e. $x \in \Omega$ is continuously differentiable with respect to (u,ξ), whose derivatives satisfy the following growth conditions

$$|D_\xi f(x,u,\xi)| + |D_u f(x,u,\xi)| \le c\big[1 + |u|^{q-1} + |\xi|^{p-1}\big], \tag{7.20}$$

for some $p, q > 1$ and $c > 0$. Note that if f is convex with respect to (u, ξ) and

$$|f(x,u,\xi)| \le L\big(1 + |u|^q + |\xi|^p\big),$$

then (7.20) is automatically satisfied, cf. [34, Lemma 2.1].

We consider a variational solution with $u \in L^q(\Omega_T, \mathbb{R}^N)$. In the variational inequality (2.3) we use the testing function $v \equiv u + s\varphi$, with $|s| \in (0,1)$ and $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. Here we recall that the variational solution obtained in Theorem 2.3 satisfies $u \in C^0([0,T]; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$ and $\partial_t u \in L^2(\Omega \times (\varepsilon, T), \mathbb{R}^N)$ for any $\varepsilon > 0$. The resulting inequality is divided by s. Afterwards, we let $s \to 0$, which amounts in taking the derivative of the mapping

$$(-1,1) \ni s \mapsto \iint_{\Omega_T} f(x, u + s\varphi, Du + sD\varphi)\, \mathrm{d}x\mathrm{d}t$$

in $s = 0$. The result is that

$$\iint_{\Omega_T} \Big[u \cdot \partial_t \varphi - D_\xi f(x,u,Du) \cdot D\varphi - D_u f(x,u,Du) \cdot \varphi\Big]\, \mathrm{d}x\mathrm{d}t = 0 \tag{7.21}$$

holds true for any $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. Consequently, the variational solution solves the associated parabolic system and therefore is a global solution to the Cauchy-Dirichlet problem associated to (7.19). In order to guarantee the existence of variational solutions with $u \in L^q(\Omega_T, \mathbb{R}^N)$, we strengthen assumption $(2.2)_1$ to

$$\mathbf{F}[u] \ge \nu\big(\|Du\|^p_{L^p(\Omega,\mathbb{R}^N)} + \|u\|^q_{L^q(\Omega,\mathbb{R}^N)}\big) - L \tag{7.22}$$

for all $u \in u_* + W_0^{1,p}(\Omega, \mathbb{R}^N) \cap L^q(\Omega, \mathbb{R}^N)$, where $\nu, L > 0$ are given constants. We note that due to Poincaré's inequality, assumption $(2.2)_1$ implies (7.22) with $q = p$ and a smaller constant $\nu > 0$. Examples of integral functionals satisfying (7.22) with $p = 2$ and $q > 2$ are given in (5.5).

Our existence result from Theorem 7.3 continues to hold under assumption (7.22) instead of $(2.2)_1$ because we may apply Theorem 7.3 to $\mathbf{F} + L$. However, the assumption (7.22) ensures that the variational solutions constructed in Theorem 7.3 additionally satisfy $u \in L^q(\Omega_T, \mathbb{R}^N)$. Therefore, the following result holds.

Theorem 7.5. *Let $\mathbf{F}$ be defined by (1.1) with a Carathéodory-function $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ that is continuously differentiable with respect to (u, ξ) for a.e. $x \in \Omega$. Suppose that the lateral boundary values u_* satisfy (2.1), that (7.20) holds true for some $p, q > 1$, and that $\mathbf{F}$ is convex on $L^2(\Omega, \mathbb{R}^N) \cap u_* + W_0^{1,p}(\Omega, \mathbb{R}^N)$ and fulfills $(2.2)_{2,3}$ and (7.22). Then, for any initial datum $u_o \in L^2(\Omega, \mathbb{R}^N)$ there exists at least one weak solution to the parabolic Cauchy-Dirichlet problem associated to (7.19), and this solution satisfies $u \in L^q(\Omega_T, \mathbb{R}^N)$.*

Therefore, under the growth assumption (7.20), every variational solution is a weak solution, too. Also a reverse result holds.

Theorem 7.6. *Under the assumptions of Theorem 7.5 any weak solution $u \in C^0([0,T]; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$ to the parabolic Cauchy-Dirichlet problem associated to (7.19) that satisfies $u \in L^q(\Omega_T, \mathbb{R}^N)$ is a variational solution in the sense of Definition 2.1.*

Proof. Let $u \in C^0([0,T]; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; u_* + W_0^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution of the parabolic Cauchy-Dirichlet problem associated to (7.19). We consider $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$. For $t \in [0,T]$ and $s \in \mathbb{R}$ we define

$$g(s,t) := \mathbf{F}\big(u(t) + s\varphi(t)\big) = \int_\Omega f\big(x, u(x,t) + s\varphi(x,t), Du(x,t) + sD\varphi(x,t)\big)\, \mathrm{d}x.$$

By the convexity of $\mathbf{F}$ we know that for a.e. $t \in [0,T]$ the function $s \mapsto g(s,t)$ is convex. Therefore, we have

$$g(1,t) \geq g(0,t) + g'(0,t), \tag{7.23}$$

for a.e. $t \in [0,T]$ if $g'(0,t)$ exists. In fact, $g'(0,t)$ exists due to the growth assumptions in (7.20) and with the same computations as before we have

$$g'(0,t) = \int_\Omega \Big[D_u f(x,u,Du)\cdot\varphi + D_\xi f(x,u,Du)\cdot D\varphi\Big]\,\mathrm{d}x,$$

for a.e. $t \in [0,T]$. Integrating both sides of (7.23) with respect to $t \in [0,T]$, we obtain

$$\int_0^T \mathbf{F}(u+\varphi)\,\mathrm{d}t \geq \int_0^T \mathbf{F}(u)\,\mathrm{d}t + \iint_{\Omega_T} \Big[D_u f(x,u,Du)\cdot\varphi + D_\xi f(x,u,Du)\cdot D\varphi\Big]\,\mathrm{d}x\mathrm{d}t.$$

Since u is a weak solution we get from the weak form (7.21) that

$$\int_0^T \mathbf{F}(u+\varphi)\,\mathrm{d}t \geq \int_0^T \mathbf{F}(u)\,\mathrm{d}t + \iint_{\Omega_T} u\cdot\partial_t\varphi\,\mathrm{d}x\mathrm{d}t.$$

This implies that u is a variational solution. In fact, with v as in Definition 2.1 we choose $\varphi = \chi_\varepsilon(t)(v-u)$, where $\chi_\varepsilon\colon [0,T]\to[0,1]$ is a cut-off function in time with $\chi_\varepsilon(0) = 0 = \chi_\varepsilon(T)$ and $\chi_\varepsilon \to \chi_{[0,\tau]}$ as $\varepsilon \downarrow 0$, for $\tau \in [0,T]$. Note that due to the differential equation (7.19) and the growth conditions (7.20), the time derivative $\partial_t u$ is a sum of a term in $L^{p'}(0,T;W^{-1,p'}(\Omega,\mathbb{R}^N))$ and one in $L^{q'}(\Omega_T,\mathbb{R}^N)$. The passage to the limit $\varepsilon\downarrow 0$ can be realized by the convexity of $\mathbf{F}$, since $0 \leq \chi_\varepsilon \leq 1$ and

$$\begin{aligned}
\int_0^\tau &\big[\chi_\varepsilon \mathbf{F}(v) + (1-\chi_\varepsilon)\mathbf{F}(u)\big]\,\mathrm{d}t \\
&\geq \int_0^\tau \mathbf{F}\big(u+\chi_\varepsilon(v-u)\big)\,\mathrm{d}t \\
&\geq \int_0^\tau \mathbf{F}(u)\,\mathrm{d}t + \iint_{\Omega_\tau} u\cdot\partial_t\big(\chi_\varepsilon(v-u)\big)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \mathbf{F}(u)\,\mathrm{d}t - \iint_{\Omega_\tau} \Big[\tfrac12 \chi_\varepsilon'|v-u|^2 + \chi_\varepsilon\partial_t v\cdot(v-u)\Big]\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for any $\tau \in [0,T]$. At this point the variational inequality (2.3) follows as $\varepsilon\downarrow 0$. □

References

[1] E. Acerbi and N. Fusco. Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.* 86(2):125–145, 1984.

[2] L. Ambrosio. Minimizing movements. (Italian) *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)* 19:191–246, 1995.

[3] L. Ambrosio, N. Gigli and G. Savaré. Gradient flows in metric spaces and in the space of probability measures. Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.

[4] J.M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.* 63:337-403, 1977.

[5] S. Boccia. Schauder estimates for solutions of higer-order parabolic systems. *Methods Appl. of Analysis*, 20: 47-68, 2013.

[6] S. Boccia and N.V. Krylov. On the fundamental matrix solution for higher-order parabolic systems. (English summary). Proceedings of the St. Petersburg Mathematical Society. Vol. XV. Advances in mathematical analysis of partial differential equations, 11–33, Amer. Math. Soc. Transl. Ser. 2, 232, Amer. Math. Soc., Providence, RI, 2014.

[7] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic equations with p,q-growth. *J. Math. Pures Appl. (9)* 100(4):535–563, 2013.

[8] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic systems with p,q-growth: a variational approach. *Arch. Ration. Mech. Anal.* 210(1):219–267, 2013.

[9] V. Bögelein, F. Duzaar, and P. Marcellini. Existence of evolutionary variational solutions via the calculus of variations. *J. Differential Equations* 256:3912–3942, 2014.

[10] V. Bögelein, F. Duzaar, and P. Marcellini. A time dependent variational approach to image restoration. *SIAM J. Imaging Sci.* 8(2):968–1006, 2015.
[11] V. Bögelein, F. Duzaar, P. Marcellini, and C. Scheven. Doubly nonlinear equations of porous medium type. *Arch. Ration. Mech. Anal.* 229(2):503–545, 2018.
[12] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* (French) North-Holland Mathematics Studies, No. 5. Notas de Matemtica (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
[13] B. Dacorogna. Convexity of certain integrals of the calculus of variations. *Proc. Roy. Soc. of Edinburgh* 107A:15–26, 1987.
[14] B. Dacorogna. Direct methods in the calculus of variations. Springer-Verlag. First edition (1989) and Second edition (2007).
[15] B. Dacorogna, Convexity of certain integrals of the calculus of variations II, *Annali di Ferrara* 36:53–62, 1991.
[16] B. Dacorogna and O. Dosly. Convexity of certain integrals of the calculus of variations III. *Boll. Unione Mat. Italiana*, 11B:837–850, 1997.
[17] S. Daneri and G. Savaré. Eulerian calculus for the displacement convexity in the Wasserstein distance. *SIAM J. Math. Anal.* 40(3):1104–1122, 2008.
[18] L.C. Evans. Quasiconvexity and partial regularity in the calculus of variations. *Arch. Rational Mech. Anal.* 95(3):227–252, 1986.
[19] L.C. Evans. Monotonicity formulae for variational problems. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 371:20120339, 2013.
[20] L.C. Evans, O. Savin, and W. Gangbo. Diffeomorphisms and nonlinear heat flows. *SIAM J. Math. Anal.* 37(3):737–751, 2005.
[21] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.
[22] C. Gallarati and M. Veraar. Evolution families and maximal regularity for systems of parabolic equations. *Adv. Differential Equations*, 22: 169-190, 2017.
[23] M. Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.
[24] E. Giusti. *Direct Methods in the Calculus of Variations*. World Scientific Publishing Company, Tuck Link, Singapore, 2003.
[25] Hongjie Dong and Doyoon Kim. Parabolic and elliptic systems with VMO coefficients. *Methods Appl. Anal.*, 16: 365-388, 2009.
[26] Hongjie Dong and Doyoon Kim. On the L_p-Solvability of Higher Order Parabolic and Elliptic Systems with BMO Coefficients. *Arch. Rational Mech. Anal.*, 199: 889–941, 2011.
[27] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3):411–435, 2006.
[28] N.V. Krylov. Parabolic and elliptic equations with VMO coefficients. *Commun. Partial Differ. Eq.*, 32: 453-475, 2007.
[29] N.V. Krylov. Parabolic equations with VMO coefficients in spaces with mixed norms. *J. Funct. Anal.*, 250: 521-558, 2007.
[30] A. Lichnewsky and R. Temam, Pseudosolutions of the time-dependent minimal surface problem. *J. Differential Equations*, 30(3):340–364, 1978.
[31] J.-L. Lions. *Quelques méthodes de rsolution des problmes aux limites non linéaires.* (French) Dunod; Gauthier-Villars, Paris 1969.
[32] P. Marcellini, Quasiconvex quadratic forms in two dimensions. *Appl. Math. Optim.*, 11:183-189, 1984.
[33] P. Marcellini. Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. *Manuscripta Math.* 51(1-3):1–28, 1985.
[34] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Rational Mech. Analysis* 105(3):267–284, 1989.
[35] A. Mielke and U. Stefanelli. Weighted energy-dissipation functionals for gradient flows. *ESAIM Control Optim. Calc. Var.* 17(1):52–85, 2011.
[36] C.B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.* 2:25–53, 1952.
[37] C.B. Morrey. Second order elliptic systems of differential equations. *Ann. Math. Studies* 33:101–159, Princeton University Press, 1954.
[38] C.B. Morrey. Multiple integrals in the calculus of variations. Springer-Verlag, Berlin, 1966.
[39] S. Müller, M.O. Rieger, V. Šverăk. Parabolic systems with nowhere smooth solutions. *Arch. Ration. Mech. Anal.* 177(1):1–20, 2005.
[40] R. Rossi, G. Savaré, A. Segatti, and U. Stefanelli. Weighted Energy-Dissipation principle for gradient flows in metric spaces. *J. Math. Pures Appl.*, to appear.
[41] D. Serre. Relations d'ordre entre formes quadratiques en compacite par compensation. *C.R. Acad. Sc. Paris* 292:785–787, 1981.

[42] F. Terpstra. Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung, *Math. Ann.* 116:166–180, 1939.

[43] W. Wieser. Parabolic Q-minima and minimal solutions to variational flow. *Manuscripta Math.*, 59(1):63–107, 1987.

VERENA BÖGELEIN, FACHBEREICH MATHEMATIK, UNIVERSITÄT SALZBURG, HELLBRUNNER STR. 34, 5020 SALZBURG, AUSTRIA
E-mail address: verena.boegelein@sbg.ac.at

BERNARD DACOROGNA, DÉPARTEMENT DE MATHÉMATIQUES, EPFL, 1015 LAUSANNE, SWITZERLAND
E-mail address: bernard.dacorogna@epfl.ch

FRANK DUZAAR, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, CAUERSTRASSE 11, 91058 ERLANGEN, GERMANY
E-mail address: frank.duzaar@fau.de

PAOLO MARCELLINI, DIPARTIMENTO DI MATEMATICA E INFORMATICA "U.DINI", UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67/A , 50134 FIRENZE, ITALY
E-mail address: paolo.marcellini@unifi.it

CHRISTOPH SCHEVEN, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STRASSE 9, 45127 ESSEN, GERMANY
E-mail address: christoph.scheven@uni-due.de